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HISTORY

★van der Waerden, B. L. Science awakening. English translation by Arnold Dresden. P. Noordhoff Ltd., Groningen, 1954. 306 pp. (28 plates). $5.00; 19 florins.

Translation, with some modifications and additions, of the author's "Ontwakende wetenschap" [Noordhoff, Groningen, 1950; these Rev. 12, 381]. Numerous illustrations have been added for this edition.

Hadamard, J. History of science and psychology of invention. Mathematika 1, 1–3 (1954).

Hadwiger, H. Der Inhaltsbegriff, seine Begründung und Wandlung in älterer und neuerer Zeit. Mitt. Naturforsch. Ges. Bern (N.F.) 11, 13–41 (1954).

Clagett, Marshall. King Alfred and the Elements of Euclid. Isis 45, 269–277 (1954).

★Ore, Øystein. Niels Henrik Abel. Et geni og hans samtid. [Niels Henrik Abel. A genius and his times.] Gyldendal Norsk Forlag, Oslo, 1954. 317 pp. (16 plates).

Blaschke, Wilhelm. Luigi Bianchi e la geometria differenziale. Ann. Scuola Norm. Super. Pisa (3) 8, 43–52 (1954).

Lecture given at Pisa on 26 September 1953.

Boyer, C. B. Carnot and the concept of deviation. Amer. Math. Monthly 61, 459–463 (1954).

★The geometry of René Descartes. (With a facsimile of the first edition, 1637.) Translated by David Eugene Smith and Marcia L. Latham. Dover Publications, Inc., New York, N. Y., 1954. ix+244 pp. Cloth $2.95; paper $1.50.

Reprint by photo-offset of the edition of 1925 [Open Court, Chicago].

★Dini, Ulisse. Opere. Vol. II. Funzioni di variabile reale e sviluppi in serie—problema di Dini-Neumann—funzioni analitiche. Edizioni Cremonese, Roma, 1954. 509 pp. 4500 Lire.

In this volume of the planned three volumes [see these Rev. 15, 383 for vol. I] papers are grouped as follows: papers on functions of a real variable; papers on expansions in series of sphere functions; papers on other series expansions; papers concerning the potential function and boundary problems for $\Delta_2 u = F$; papers on the theory of analytic functions. The first three groups are provided with introductions by Sansone and Scorza-Dragoni, the fourth by Picone, and the last by Cecioni.

★Eulerus, Leonhardus. Opera omnia. Series prima. Opera mathematica. Vol. XXVII. Commentationes geometricae. Vol. secundum. Edidit Andreas Speiser. Societatis Scientiarum Naturalium Helveticae, Lausanne, 1954. xlvii+400 pp.

This second geometrical volume, like the first [these Rev. 15, 770], has an illuminating introduction by A. Speiser. Apart from two papers (in French) on spherical and spheroidal trigonometry, almost all are concerned with plane curves. Some of the problems considered are as follows: to find a one-parameter family of curves whose orthogonal trajectories are their images by reflection in the y-axis; to find curves, other than conics, that possess "diameters" bisecting systems of parallel chords; to find curves by means of which light-rays from a certain point will return there after two reflections; to find curves that are similar to their kth evolutes; to find curves, other than the parabola, that have perpendicular tangents at the ends of chords through a certain point. There is also a paper about the various kinds of cusp that may occur on an algebraic curve; and one about subnormals, leading to the functional equation $f(x+f(x)) = nf(x)$. *H. S. M. Coxeter.*

★Levi-Civita, Tullio. Opere matematiche. Memorie e note. Vol. I. 1893–1900. Pubblicate a cura dell'Accademia Nazionale dei Lincei. Nicola Zanichelli Editore, Bologna, 1954. xxx+563 pp. (1 plate). 8000 lire.

This volume contains Levi-Civita's papers from the years 1893–1900 and a commemorative address by U. Amaldi.

★Gerasimova, V. M. Ukazatel' literatury po geometrii Lobačevskogo i razvitiyu ee ideĭ. [Index to the literature of the geometry of Lobačevskiĭ and the development of his ideas.] Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1952. 192 pp. 6 rubles.

For the most part (pp. 11–151) this book contains a list, ordered by year, of publications on Lobačevskian and elliptic geometry, on non-Euclidean geometry in so far as it pertains to these, and on the life and work of Lobačevskiĭ. The list goes from 1823–1952. There is also a subject index (pp. 152–166), an author index (pp. 167–175), and data on the Lobačevskiĭ prize of the Kazan Physical-Mathematical Society.

★Œuvres de Henri Poincaré. Publiées sous les auspices de l'Académie des Sciences par la Section de Géométrie. Tome VI. Publié avec la collaboration de René Garnier et Jean Leray. Gauthier-Villars, Paris, 1953. 540 pp. 7000 fr.

This volume contains Poincaré's papers on geometry and topology.

Have

*Dantzig, Tobias. **Henri Poincaré, critic of crisis. Reflections on his universe of discourse.** Charles Scribner's Sons, New York-London, 1954. xi+149 pp.

Cassina, Ugo. Giovanni Vacca. La vita e le opere. Ist. Lombardo Sci. Lett. Rend. Parte Generale e Atti Ufficiali (3) 17(86), 185–200 (1 plate) (1953).

A list of Vacca's published work is included. Vacca was born 18 November 1872 and died 6 January 1953.

On order

*Volterra, Vito. **Opere matematiche. Memorie e note. Vol. I. 1881–1892.** Accademia Nazionale dei Lincei, Roma, 1954. xxxiii+604 pp. (1 plate). 8000 lire.

This volume contains commemorative addresses by G. Castelnuovo and C. Somigliano, a biographical sketch by J. Pérès and Volterra's papers from the years 1881–1892.

*Voronoĭ, G. F. **Sobranie sočineniĭ v treh tomah. [Collected works in three volumes.]** Izdatel'stvo Akademii Nauk Ukrainskoĭ SSR, Kiev. Vol. I, 1952, 399 pp. (1 plate); Vol. II, 1952, 391 pp. (2 plates); Vol. III, 1953, 306 pp. (3 plates). 28.70, 28.80, 20.90 rubles.

Comments on individual papers have been written by B. A. Venkov, B. N. Delone, Yu. V. Linnik, A. A. Kiselev, I. R. Šafarevič, I. B. Pogrebysskiĭ, and N. G. Čudakov. An essay on Voronoĭ's life and scientific work by Štokalo and Pogrebysskiĭ is included in vol. III. Voronoĭ was born in 1868 and died in 1908.

Artobolevskiĭ, I. I. The works of N. E. Žukovskiĭ on applied mechanics. Akad. Nauk SSSR. Trudy Sem. Teorii Mašin i Mehanizmov 12, no. 46, 5–14 (1 plate) (1952). (Russian)

FOUNDATIONS

Fitch, Frederic B. A definition of negation in extended basic logic. J. Symbolic Logic 19, 29–36 (1954).

In this paper the system of extended basic logic **K'** [same J. 13, 95–106 (1948); these Rev. 9, 559] is simplified as follows. The proper ancestral of a relation is shown to be definable in terms of the other concepts of the system. Also the dual of the proper ancestral and negation are shown definable. The rules needed to define **K'** here constitute in fact a proper sub-set of the rules used to define **K'** originally.

R. M. Martin (Philadelphia, Pa.).

Ridder, J. Über modale Aussagenlogiken und ihren Zusammenhang mit Strukturen. IV^bis. Nederl. Akad. Wetensch. Proc. Ser. A. 56 = Indagationes Math. 15, 378–388 (1953).

Ridder, J. Über modale Aussagenlogiken und ihren Zusammenhang mit Strukturen. V. Nederl. Akad. Wetensch. Proc. Ser. A. 57 = Indagationes Math. 16, 2–8 (1954).

Ridder, J. Über modale Aussagenlogiken und ihren Zusammenhang mit Strukturen. VI. Nederl. Akad. Wetensch. Proc. Ser. A. 57 = Indagationes Math. 16, 117–128 (1954).

In continuation of part IV [same Proc. 56, 99–110 (1953); these Rev. 15, 90], the author presents decision procedures for various propositional calculi: I, an intuitionistic calculus, M a minimal calculus, and A_2, a subsystem of the minimal calculus, also for modal calculi, two corresponding to each of the previously mentioned systems, and finally for systems dual to each of these. The methods are similar to those used in the earlier portions of the work. In Part VI, the author establishes results similar to those of McKinsey and Tarski [Ann. of Math. (2) 47, 122–162 (1946), Theorems 1.14 and 1.15; these Rev. 7, 359] which relate Brouwerian algebras to closure algebras. Here the analogous results are shown between structures corresponding to the various subsystems of the propositional calculus and structures corresponding to the modal systems. Further, the author establishes relations between decision procedures for the various calculi and the corresponding modal systems similar to that observed by McKinsey and Tarski [J. Symbolic Logic 13, 1–15 (1948); these Rev. 9, 486] between decision procedures for the intuitionistic logic and Lewis's system *S*4.

D. Nelson (Washington, D. C.).

Brouwer, L. E. J. Intuitionistic differentiability. Nederl Akad. Wetensch. Proc. Ser. A. 57 = Indagationes Math. 16, 201–203 (1954). (Dutch)

Let $f(x)$ be a function which is defined for every real number x. The author gives definitions, applicable in intuitionistic mathematics, of a weak derivative and of a strong derivative of $f(x)$ in a point P; the former is a negative notion, the latter the corresponding positive one. From the classical point of view the two definitions would be equivalent, but this is not true intuitionistically, as is seen from an example of a function which possesses a weak derivative at $x=0$, while it cannot possess a strong derivative at $x=0$. In an earlier paper [same Proc. 57, 109–111 (1954); these Rev. 15, 925] it was shown by the author that a certain function $g(x)$ is almost everywhere weakly derivable; here he shows that $g(x)$ is almost everywhere strongly derivable.

A. Heyting (Amsterdam).

Brouwer, L. E. J. An example of contradictority in classical theory of functions. Nederl. Akad. Wetensch. Proc. Ser. A. 57 = Indagationes Math. 16, 204–205 (1954).

It is shown that, intuitionistically, the assertion that each monotone full function, defined everywhere in [0, 1], must be differentiable for some value of the independent variable, is contradictory.

A. Heyting (Amsterdam).

Esenin-Vol'pin, A. S. The unprovability of Suslin's hypothesis without the axiom of choice in the system of axioms of Bernays-Mostowski. Doklady Akad. Nauk SSSR (N.S.) 96, 9–12 (1954). (Russian)

This paper has reference to an axiom-system $\mathfrak{S}$ for set-theory proposed by Mostowski [Fund. Math. 32, 201–252 (1939), pp. 201–208]. This system $\mathfrak{S}$ is a modification of the system of Bernays [J. Symbolic Logic 2, 65–77 (1937); 6, 1–17 (1941); 7, 65–89, 133–145 (1942); 8, 89–106 (1943); 13, 65–79 (1948); these Rev. 2, 210; 3, 290; 4, 183; 5, 198; 10, 3]; it is similar to that of Gödel [Consistency of the continuum hypothesis, Princeton, 1940; these Rev. 2, 66]. The principal difference between Mostowski's system and those of Bernays and Gödel is that the former admits individuals (i.e. elements which are not sets); it also does not contain the axiom of choice, and was constructed for the specific purpose of showing the independence of the latter. The hypothesis of Suslin is the statement that if an

ordered set is such that every non-overlapping set of open intervals is at most denumerable, then the ordered set has a denumerable subset. The thesis of this paper is to exhibit a model Z which satisfies the axiom of $\mathfrak{S}$ and contains an ordered subset (viz. the individuals) such that every set of non-overlapping intervals is denumerable but there is no dense denumerable subset. The paper describes the construction of Z and gives an outline of the proof that it has the properties stated. Putting in all the details of this proof would be a long and tedious process, and the reviewer has not attempted it; but it seems to be fairly straightforward. It also appears that the proof involves the axiom of choice at one point. The author points out that the axiom of choice is not valid for Z; hence nothing is shown about independence if the axiom of choice is assumed.

H. B. Curry (State College, Pa.).

***Finsler, Paul. Ueber die Berechtigung infinitesimalgeometrischer Betrachtungen.** Convegno Internazionale di Geometria Differenziale, Italia, 1953, pp. 8–12. Edizioni Cremonese, Roma, 1954. 4000 Lire.

The paper is not primarily concerned with differential geometry, but with the foundations of set theory. The content is much the same as that of a later paper [Elemente der Math. **9**, 29–35 (1953); these Rev. **15**, 670].

G. Kreisel (Reading).

***Hadamard, Jacques. An essay on the psychology of invention in the mathematical field.** Dover Publications, Inc., New York, 1954. xiii+145 pp. \$1.25; clothbound \$2.50.

Reprint by photo-offset of the revised edition of 1949 [Princeton Univ. Press; these Rev. **10**, 423] and not the first edition of 1945 as is stated on back of the title page.

ALGEBRA

Katz, Leo, and Powell, James H. The number of locally restricted directed graphs. Proc. Amer. Math. Soc. **5**, 621–626 (1954).

The enumeration is of graphs with n nodes and t directed branches such that at node P_i there are r_i branches leaving and s_i branches entering; $\sum r_i = \sum s_i = t$. This is the same as the number of ways of putting 0 or 1 in each cell of a chessboard so that the row sums are the r_i and the column sums the s_i, and the main diagonal contains 0's only. Without the last, the problem has been "solved" by MacMahon [Combinatory analysis, v. I, Cambridge, 1915, p. 224 et seq.] and by P. V. Sukhatme [Philos. Trans. Roy. Soc. London. Ser. A. **237**, 375–409 (1938)]. The authors solve their problem by relating it to Sukhatme's through the use of an operator of Hammond type defined by

$$\delta_i f(r_1, \cdots, r_i, \cdots, r_n; s_1, \cdots, s_i, \cdots, s_n) = f(r_1, \cdots, r_i - 1, \cdots, r_n; s_1, \cdots, s_i - 1, \cdots, s_n).$$

J. Riordan (New York, N. Y.).

Braumann, Pedro Bruno Teodoro. Beziehungen zwischen Kombinationen und Partitionen. Univ. Lisboa. Revista Fac. Ci. A. Ci. Mat. (2) **3**, 75–76 (1954).

Expressions for the sums of the combinations of the numbers $1, 2, \cdots, l-1$. *O. Ore* (New Haven, Conn.).

Braumann, Pedro. Bemerkungen zu einer aus der Kombinatorik bekannten Formel. Univ. Lisboa. Revista Fac. Ci. A. Ci. Mat. (2) **3**, 158–160 (1954).

Expressions for the moments of the sum of independent variables derived by the multinomial theorem.

O. Ore (New Haven, Conn.).

Gouarné, René. Sur une généralisation des polynomes d'Hermite en relation avec le dénombrement des permutations de n objets ne présentant pas de cycles d'ordre supérieur à un entier donné p. C. R. Acad. Sci. Paris **239**, 470–472 (1954).

The generalized polynomials, $G(n, p) = G(n, p; x)$, are taken as defined by the recurrence

$$G(n, p) = xG(n-1, p) + (n-1)G(n-2, p) + \cdots + (n-1)\cdots(n-p+1)G(n-p, p).$$

(Boundary values are not mentioned but are the natural ones: $G(-n, p) = 0$, $G(0, p) = 1$, so that $G(n, 1) = x^n$, $G(n, 2) = (-i)^n H_n(ix)$ with H_n an Hermite polynomial.) It is shown that: (1) $DG(n, p) = nG(n-1, p)$, $D = d/dx$, the property for Appell polynomials;

$$(2) \qquad ny = xy' + y'' + \cdots + d^p y/dx^p, \quad y = G(n, p);$$

(3) $g(n, p) = G(n, p; 1)$ is the number of permutations of n elements for which no cycle is longer than p; and (4) their exponential generating function is

$$\sum_0^\infty G(n, p)u^n/n! = \exp\,(xu + u^2/2 + \cdots + u^p/p).$$

But the last is a special case of results given by Touchard [Acta. Math. **70**, 243–297 (1939)] in his study of cycles in permutations. *J. Riordan* (New York, N. Y.).

Fjeldstad, Jonas Ekman. A generalization of Dixon's formula. Math. Scand. **2**, 46–48 (1954).

The formula of Dixon in question is that relating to the sum of cubes of binomial coefficients. The generalization is given in two forms, of which the simpler is the symmetric one

$$\sum (-1)^s \binom{m+n}{m+s}\binom{n+p}{n+s}\binom{p+m}{p+s} = \frac{(m+n+p)!}{m!\,n!\,p!}.$$

The sum is of a finite number of terms, determined naturally by the binomial coefficients. Proof is by alternate evaluations of an integral, but can also be obtained by a limiting process from Dixon's theorem [Erdélyi et al., Higher transcendental functions, vol. 1, McGraw-Hill, New York, 1953, p. 189; these Rev. **15**, 419]. *J. Riordan.*

Zelen, Marvin. Analysis for some partially balanced incomplete block designs having a missing block. Biometrics **10**, 273–281 (1954).

This paper outlines the intra-block analysis (if a whole block is lost) for partially balanced incomplete block designs with two associate classes such that all treatments in the missing block are the same associates of each other.

H. B. Mann (Columbus, Ohio).

Gardner, Robert S. The analysis of a replicated balanced incomplete block design. U. S. Naval Ordnance Test Station, Inyokern, Calif., Tech. Memo. **967**, iv+15 pp. (1953).

Formulas for the analysis of variance of the designs in the title are explicitly given. Problems of point and interval estimation of parameters are likewise discussed.

H. B. Mann (Columbus, Ohio).

Kerawala, Sulaiman. Symmetrical incomplete block designs with $\lambda=2$. Scientist, Pakistan 1, nos. 2 and 3, 1–24 (1953).

The author enumerates the non-isomorphic solutions of the designs in the title with $k=3$, 4, 5 and 6. He does not seem to be acquainted with the results of Chowla and Ryser [Canadian J. Math. **2**, 93–99 (1950); these Rev. **11**, 306] and gives a long enumeration proof for the non-existence of designs with $v=b=29$, $r=k=8$, $\lambda=2$. He also fails to give credit to Schützenberger for his result on symmetrical incomplete balanced block designs with even v [Ann. Eugenics **14**, 286–287 (1949); these Rev. **11**, 3].

H. B. Mann (Columbus, Ohio).

Hall, Marshall, and Ryser, H. J. Normal completions of incidence matrices. Amer. J. Math. **76**, 581–589 (1954).

Cette étude est consacrée aux configurations (v, k, λ) où k est compris entre les entiers positifs λ et v, avec la condition $\lambda=k(k-1)/(v-1)$ [pour les définitions, voir H. J. Ryser, même J. **74**, 769–773 (1952); ces Rev. **14**, 346]. Il est d'abord montré que, si deux vecteurs $x=(x_1, \cdots, x_n)$ et $y=(y_1, \cdots, y_n)$ satisfont à:

$$x_1^2+\cdots+x_n^2=y_1^2+\cdots+y_n^2=c\neq 0,$$

alors il existe une matrice orthogonale 0 telle que $x0=y$, dans tout champ de caractéristique $\neq 2$. Puis, que si B est congruente à l'identique par rapport au corps des nombres rationnels, il existe un C rationnel tel que $CC'=C'C=B$. Cette généralisation d'un théorème d'Albert est dérivée de cet autre: S'il existe un C rationnel tel que $C'C=B$ et $CS=kS$, où S est la matrice dont tous les éléments sont des 1, alors $CC'=B$. Enfin, si B est rationnellement congruente à l'identique, si A_1 est une matrice $r\times v$ composée avec des 0 et des 1, telle que $A_1A_1'=B_1$ où B_1 est la matrice $r\times r$ avec des k dans la diagonale principale et des λ ailleurs, alors il existe une matrice $v\times v$ rationnelle A dont les r premières lignes sont A_1 et telle que $AA'=A'A=B$. Le mémoire se termine par l'examen de l'équation $AA'=B$ où A est entier. Exemple pour $N=10$.

A. Sade.

Huff, Charles W. On pairs of matrices (of order two) A, B satisfying the condition $e^Ae^B=e^{A+B}\neq e^Be^A$. Rend. Circ. Mat. Palermo (2) **2** (1953), 326–330 (1954).

All matrices of order 2 which satisfy the equation of the title are found. The analysis starts from the fact that e^A is a linear combination of I and A. The matrices found are the same as those described in the following review.

J. L. Brenner (Aberdeen, Md.).

Čebotarev, G. N. On the solution of the matrix equation $e^B\cdot e^C=e^{B+C}$. Doklady Akad. Nauk SSSR (N.S.) **96**, 1109–1112 (1954). (Russian)

The complete solution of the equation of the title, when B, C have order 2, is as follows. (I) B, C commutative. (II) B, C not commutative, but e^B, e^C, e^{B+C} scalar. This happens if the differences of the characteristic roots of B, C, $B+C$ are nonzero multiples of $2\pi i$: $\xi=2\pi ik$, $\eta=2\pi il$, $\zeta=2\pi im$, and $k+l+m$ is even, where ξ is the difference of the characteristic roots of B, etc.; and k, l, m are integers. [This case was considered by Fréchet, Rend. Circ. Mat. Palermo (2) **1**, 11–27 (1952); these Rev. **14**, 237.] (III) e^B, e^C not commutative, $BC=-\lambda\mu I+\mu B+\lambda C$, where μ is a characteristic root of C and λ is a characteristic root of B, and the relations

$$(e^\xi-1)/\xi=(e^\eta-1)/\eta=(e^{-\xi}-1)/-\xi=(e^{-\eta}-1)/-\eta=1;$$
$$(e^\xi-1)/\xi=1;\quad (e^\eta-1)/\eta=1$$

hold in the respective cases $\xi\neq 0$, $\eta\neq 0$, $\xi\neq\eta$; $\xi=\eta\neq 0$; $\xi\neq 0$, $\eta=0$; $\xi=0$, $\eta\neq 0$. The following lemma holds for matrices of any order. The relation $BC=\gamma E+\mu B+\lambda C$ can hold for non-commuting matrices only if λ is a characteristic root of B, μ is a characteristic root of C, and the relation $-\gamma=\lambda\mu$ holds. Applications to the problem of simultaneous conformal maps with special boundary values are given. Several examples are exhibited.

J. L. Brenner.

Kakar, A. G. Non-commuting solutions of the matrix equation $\exp(X+Y)=\exp X\exp Y$. Rend. Circ. Mat. Palermo (2) **2** (1953), 331–345 (1954).

For any order n, the author finds matrices X, Y which are not permutable and which satisfy the equation of the title. Vectors r, s are found so that for some values of the integral parameters a_i, $[X, Y]$ can be $[\mathrm{diag}\,(2\pi ia_1, \cdots, 2\pi ia_n), rs^*]$. In this case, the pair X, Y is irreducible; the relation $\exp X\exp Y=\exp Y\exp X$ holds. In a second chapter, solutions are found for which this last relation does not hold; these solutions are reducible (in fact, triangular) but not completely reducible.

J. L. Brenner (Aberdeen, Md.).

Gluskin, L. M. An associative system of square matrices. Doklady Akad. Nauk SSSR (N.S.) **97**, 17–20 (1954). (Russian)

The following are conditions that a system G with an associative operation should be isomorphic with the system G_n of all square matrices of order n (under multiplication). Let $0, e_1, e_2, \cdots, e_n$ be permutable primitive idempotents; let e_1Ge_1 be isomorphic with the multiplicative system of the underlying field. For every k, let there be two elements x_k, x_k' (in Ge_k) with product e_1. Let every primitive idempotent have a non-trivial (left-) annihilator. For every set x_{ik} of n^2 elements, where x_{ik} is in e_iGe_k, let there be one (and only one) element x such that the relations $e_ixe_k=x_{ik}$ hold.

J. L. Brenner (Aberdeen, Md.).

Roy, S. N. A useful theorem in matrix theory. Proc. Amer. Math. Soc. **5**, 635–638 (1954).

If A and B are any two matrices, the theorem of this article states that the square of the absolute value of any characteristic root λ of AB is greater [less] than or equal to the product of the minimum [maximum] characteristic root of AA^* by the minimum [maximum] characteristic root of BB^*. A short proof is the following. First, if F is hermitian, and x is a variable unit vector, $xx^*=1$, the minimum of xFx^* is equal to the minimum characteristic root ρ_1 of F.

$$(F=U\cdot\mathrm{diag}\,(\rho_1, \cdots, \rho_n)\cdot U^*,$$
$$UU^*=I\Rightarrow xFx^*=y_1\rho_1\bar{y}_1+\cdots+y_n\rho_n\bar{y}_n$$

with $y=xU$, $yy^*=1$.) Second, the minimum characteristic root of $(AB)(AB)^*$ is less than or equal to the square of the absolute value of any characteristic root λ of AB. ($zz^*=1$, $z(AB)=\lambda z\Rightarrow z(AB)(AB)^*z^*=\lambda\bar{\lambda}$; apply first statement with $F=(AB)(AB)^*$.) Third, the characteristic roots of $(AB)(AB)^*$ and A^*ABB^* are the same. Fourth, the minimum characteristic root of the product of two hermitian matrices, as A^*A, BB^*, is greater than or equal to the product of the respective minimum characteristic roots of the factors.

J. L. Brenner (Aberdeen, Md.).

Brenner, J. L. Orthogonal matrices of modular polynomials. Duke Math. J. **21**, 225–231 (1954).

The main part of this paper is concerned with orthogonal matrices whose elements are polynomials in a single variable

with coefficients lying in the prime field of 2 elements. The author investigates the decomposition of such a matrix A into a product of orthogonal matrices of lower dimension, i.e. matrices which are of the form T diag $(B, I)T^{-1}$, where B is of lower dimension than A and T is a permutation matrix. It is shown that for certain 4-rowed orthogonal matrices such a decomposition is impossible, whilst it can always be achieved when the dimension exceeds 4.

Some structure theorems are obtained for groups of orthogonal matrices of the type described. There are also some new results concerning the case of a prime field of p (>2) elements, leading to the construction of non-constant orthogonal matrices in 3 dimensions. [The reviewer has noticed a few inaccuracies, one of which slightly affects the proof of Lemma 3: the author has remarked that diag $(C_{r-1}, 0)$ should read diag $(C_{r+1}, 0)$; the case $r+1=n$ then requires a separate argument, which however easily follows from Lemma 2.] *W. Ledermann* (Manchester).

★**Burnside, William Snow, and Panton, Arthur William. The theory of equations with an introduction to the theory of binary algebraic forms. Vol. I.** 10th ed. reprinted. S. Chand & Co., Delhi, 1954. x+223 pp. Rupees 7/8/0.

Saban, Giacomo. Funzioni totalmente derivabili di variabili in [$(n+1)$-duali] un'algebra ad $n+1$ unità definita nel corpo reale. Giorn. Mat. Battaglini (5) 2(82), 267–276 (1954).

The author obtains necessary and sufficient conditions that a function with domain and range in an $(n+1)$-dual algebra [Spampinato, Rend. Accad. Sci. Fis. Mat. Napoli (4) 18, 219–226 (1952); these Rev. 14, 346] shall have a unique derivative. The result is extended to functions of several variables. *G. B. Huff* (Athens, Ga.).

Abstract Algebra

Herstein, I. N. A note on rings with central nilpotent elements. Proc. Amer. Math. Soc. 5, 620 (1954).

The author's previous result [Amer. J. Math. 75, 864–871 (1953); these Rev. 15, 392] is used to generalize an earlier theorem [Proc. Amer. Math. Soc. 1, 370–371 (1950); these Rev. 12, 75] as follows: Any ring, all of whose nilpotent elements are in the center, and each of whose elements generates a finite subring, is commutative.

R. D. Schafer (Storrs, Conn.).

Johnson, R. E. Semi-prime rings. Trans. Amer. Math. Soc. 76, 375–388 (1954).

A ring R is called semi-prime if the zero ideal of R is an intersection of prime ideals of the ring. It is known that a necessary and sufficient condition for a ring R to be semi-prime is that the ordinary radical of R (i.e. the sum of all nilpotent ideals of R) should be zero. The class of semi-prime rings contains as a subclass the prime rings (i.e. rings where the zero ideal is prime) which were studied by the author in previous papers [Duke Math. J. 18, 799–809 (1951); Trans. Amer. Math. Soc. 74, 351–357 (1953); these Rev. 13, 618; 14, 839] and some of the results and concepts of these papers are applied by the author, partially in a generalized form, in the present investigation.

Fundamental in the present study are the following two concepts: 1) The component I^c of a right ideal I, i.e., the left annihilator of the right annihilator of I. 2) The concept of a prime right ideal: The right ideal I is called a prime right ideal if and only if $AB\subseteq I$, A, B right ideals with $B^c=R$, implies $A\subseteq I$ (in case R is a prime ring the condition $B^c=R$ is equivalent to the condition $B\neq 0$). With every right ideal I the author associates the least prime right ideal $p(I)$ containing I. The mapping $I\to p(I)$ is a closure operation on the set of the right ideals of R. Moreover, the following result is proved. Denote by $\mathfrak{R}$ a structure of R, i.e. a system of subsets of R which is closed under the operation of infinite intersection such that for each right ideal I of R there is a least element I^* of $\mathfrak{R}$ containing I such that the following axioms are satisfied: (1) $0, R\in\mathfrak{R}$; (2) $(I\cap I')^*=I^*\cap I'^*$; (3) $(I:a)^*=(I^*:a)$, where $a\in R$ and $(I:a)$ is the set of all x such that $ax\subseteq I$. Then the set of all prime right ideals is a structure that contains every other structure of the ring. Of particular interest are semi-prime rings with a-structures (=atomic structures), i.e. structures containing minimal nonzero elements. Thus, e.g., semi-simple rings (in the sense of Jacobson) containing minimal right ideals are semi-prime rings with a-structures. For a semi-prime ring R with a-structure $\mathfrak{R}$ the author considers for all atoms I (i.e. nonzero minimal elements of $\mathfrak{R}$) the set of all components I^c which are shown to be prime rings, and defines the base B of R to be the ring union of all I^c. This is analogous to the definition of the socle of a ring, and one verifies that the base B of R contains the socle of R. With the help of this notion the author derives the following result: Let $N(R)$ be the universal extension ring of R containing R as an ideal and having the property that $R^c=N(R)$ [Johnson, Duke Math. J. 20, 569–573 (1953); these Rev. 15, 391]. Then R is a semi-prime ring with a-structure if and only if $B\oplus B'\subseteq R\subseteq N(B)\oplus N(B')$, where B is the (complete) direct sum of prime rings each having an a-structure and B' is a semi-prime ring having a non-atomic structure. *J. Levitzki* (Jerusalem).

Fuchs, L. On the fundamental theorem of commutative ideal theory. Acta Math. Acad. Sci. Hungar. 5, 95–99 (1954). (Russian summary)

The author defines: the fundamental theorem of ideal theory holds for an ideal A in a commutative ring R if $A=P_1^{k_1}\cdots P_r^{k_r}$ with uniquely determined prime ideals P_i and uniquely determined minimal exponents k_i, and if for every ideal B such that $A\subseteq B\subseteq R$, $B=P_1^{l_1}\cdots P_r^{l_r}$ with $0\leq l_i\leq k_i$. He then proves in an elementary way: the fundamental theorem of ideal theory holds for an ideal A in a commutative ring with unit if and only if (1) R/A satisfies the minimum condition and (2) if the principal component $A(P)$ of A for a prime ideal divisor P of A satisfies $A(P)\subset P$, then P^2 is an immediate multiple of P. *I. N. Herstein.*

Witt, Ernst. Treue Darstellungen beliebiger Liescher Ringe. Collectanea Math. 6, 107–114 (1953).

By a suitable modification of the methods used to establish the independence of the standard monomials in the universal associative algebra of a Lie algebra, the author proves that the natural imbedding of a Lie ring L in its universal associative ring is faithful. The technical problem stemming from the lack of a basis for L is solved by using in the additive group of L an appropriate subgroup which is a direct sum of cyclic groups. If L is a Lie ring with commutative operator domain Ω, sufficient conditions on Ω are noted in order that this method of proof be valid. A final remark cites an example which shows that some restriction on Ω is essential for the existence of a faithful imbedding.

W. G. Lister (Providence, R. I.).

Hua, Loo-Keng. A generalization of Hamiltonian matrices. Acta Sci. Sinica **2**, 1–58 (1953).

L'auteur étudie en substance, dans ce travail, les formes hermitiennes (ou antihermitiennes) sur un corps non commutatif admettant une involution et de caractéristique $\neq 2$. A cette dernière restriction près, ses résultats coïncident avec ceux d'un article du rapporteur sur le même sujet [Trans. Amer. Math. Soc. **72**, 367–385 (1952); ces Rev. **14**, 134], mais ont été obtenus indépendamment; les méthodes sont aussi substantiellement les mêmes, bien que l'auteur les exprime le plus souvent dans le langage des matrices, qui leur donne un aspect quelque peu différent.

J. Dieudonné (Evanston, Ill.).

Skopin, A. I. p-extensions of a local field containing $\sqrt[p^M]{1}$. Doklady Akad. Nauk SSSR (N.S.) **95**, 29–32 (1954). (Russian)

Soient k un corps de nombres p-adiques, p la caractéristique de son corps résiduel, n son dégré par rapport au corps p-adique rationnel. I. Chafarévitch (Šafarevič) a prouvé [Mat. Sbornik N.S. **20**(62), 351–363 (1947); ces Rev. **8**, 560], à terminologie près, que le groupe de Galois (organisé par sa topologie de Krull) $G_{\Omega_p/k}$ de la p-extension maximale Ω_p/k (autrement dit du composé de toutes les extensions galoisiennes de k, dont le degré est une puissance de p) est le complété du groupe libre de $n+1$ générateurs par rapport à sa p-topologie (autrement dit, sa topologie, où ses sous-groupes d'indice fini puissance de p forment une base de la famille des voisinages de l'unité) quand k ne contient aucune racine p-ième primitive de l'unité. La quéstion se pose de déterminer le groupe de Galois de Ω_p/k sans cette condition restrictive.

G étant un groupe topologique quelconque, soit G' l'adhérence de son sous-groupe engendré par les puissances p-ièmes et par les commutateurs de ses éléments. Posons $G^{(i+1)}=(G^{(i)})'$. Supposant que k contient une racine primitive p^m-ième de l'unité, l'auteur détermine complètement le facteur (dont l'ordre est fini) $G^*/G^{*(m)}$ de $G^*=G_{\Omega_p/k}$. Il se trouve que ce facteur est isomorphe au facteur analogue du groupe fondamental d'une surface close de genre $h=(n+2)/2$, autrement dit à celui du groupe ϕ_h de $2h$ générateurs $a_1, a_2, \cdots, a_{2h}$ avec la seule relation définissante $(a_1, a_2)(a_3, a_4)\cdots(a_{2i-1}, a_{2i})\cdots(a_{2h-1}, a_{2h})$, où (a, b) est le commutateur des a, b. L'idée de la demonstration est, d'une manière très vague, la suivante: par considération des classes de cohomologie (sous forme des invariants des k-algèbres simples normales) on démontre que $G^*/G^{*(m)}$ est une image homomorphe de ϕ_h et, ensuite, par considérations sur les ordres des groupes analogues à celles de Chafarévitch (mais où le théorème de Schreier sur le nombre des générateurs des sous-groupes d'un groupe libre est remplacé par un certain théorème connu dans la topologie algébrique), on montre que le noyau de cet homomorphisme est $\phi_h^{(m)}$.

Les résultats de ce travail se récoupent, en partie, avec ceux du travail récent (et indépendant) de Y. Kawada [voir l'analyse ci-dessous]. Kawada démontre (par des méthodes plus élémentaires, d'ailleurs, que celles du travail considéré de Skopin) que G^* est isomorphe, en tant que groupe topologique, au quotient du complété F^*_{n+2}, par rapport à sa p-topologie, du groupe libre F_{n+2} de $n+2$ générateurs par l'adhérence de son sous-groupe invariant engendré, en tant que tel, par un unique élément ρ de ce groupe. Il n'arrive pas, toutefois, à déterminer complètement la forme de G^*. Ainsi, tandis que Kawada détermine, d'une manière incomplète seulement, la structure du groupe G^* tout entier, l'auteur détermine complètement la structure d'un facteur assez petit de ce groupe.

M. Krasner (Paris).

Kawada, Yukiyosi. On the structure of the Galois group of some infinite extensions. I. J. Fac. Sci. Univ. Tokyo. Sect. I. **7**, 1–18 (1954).

The author studies the topological Galois group G of the maximal p-extension of a field k. Theorem 1: If k has characteristic p, or if k has characteristic different from p and k contains a primitive pth root of 1 and there exists no noncommutative division algebra over k of exponent p, then G is topologically isomorphic to a free topological p-group with N generators. Here N is the cardinal number of a minimal basis of, in the first case the additive group k/Pk (where Pk is the set of all elements x^p-x with $x \in k$), and in the second case the multiplicative group $k^*/(k^*)^p$. Theorem 2: If k is a $\mathfrak{p}$-adic number field containing a primitive pth root of 1, then G is topologically isomorphic to a topological p-group with $[k:R_p]+2$ generators and one fundamental relation (where R_p is the field of rational p-adic numbers). The case in which k does not contain a primitive pth root of 1 was settled by Šafarevič [Mat. Sbornik N.S. **20**(62), 351–367 (1947); these Rev. **8**, 560; **12**, 1001], who showed that in this case G is topologically isomorphic to a free topological p-group with $[k:R_p]+1$ generators.

E. R. Kolchin (Paris).

Kawada, Yukiyosi. On the structure of the Galois group of some infinite extensions. II. J. Fac. Sci. Univ. Tokyo. Sect. I. **7**, 87–106 (1954).

The author studies certain infinite normal field extensions, the structure of their Galois groups, and their cohomology theory. Typical of his many results are the determination of the Galois group of the maximal abelian extension of any field of characteristic 0 containing all roots of unity, and the results described below. Let k be a field, Ω its maximal separable extension, $K \subset \Omega$ a normal extension of k of finite degree, A_k, A_K the maximal abelian extensions of k, K respectively, G, Γ_k, Γ_K the Galois groups of K/k, A_k/k, A_K/K respectively. G operates in a natural way on K^* (multiplicative group of K) and Γ_K. Consider the following six properties of K/k. P1: $H^r(G, K^*)=0$ $(r=0, \pm 1, \cdots)$. P2: $H^2(G, K^*)=0$. P3: $H^0(G, K^*)=0$ (i.e. $k^*=N_{K/k}K^*$). Q1: A certain mapping (induced by a pairing) is an isomorphism $H^r(G, \Gamma_K)\approx H^{r-2}(G, Z)$ $(r=0, \pm 1, \cdots)$, where Z is the group of integers. Q2: (i) $H^1(G, \Gamma_K)=0$ and (ii) $H^2(G, \Gamma_K)$ is cyclic of order $[G:1]$ generated by the canonical cocycle. Q3: A certain mapping (transfer) is an isomorphism $\Gamma_k \approx$ the group of elements of Γ_K invariant under G. Also, if, for every field L between k and K, K/L has property P1, $\cdots$, or Q3, then we say that K/k has property P*1, $\cdots$, or Q*3 respectively. It is common knowledge that P*1, P*2, P*3 are equivalent, and Q*2 implies Q*1 [Tate, Ann. of Math. (2) **56**, 294–297 (1952); these Rev. **14**, 252]. Also, if k is quasi algebraically closed or an algebraic number field containing all roots of unity, then P*1 holds; and if k is an algebraic function or formal power series field in one variable over a finite coefficient field or is a $\mathfrak{p}$-adic number field, then Q*1 holds. After showing that Q1 implies Q2 and Q3, the author succeeds in proving that if k has characteristic 0 and contains all roots of unity then P3 is equivalent to Q3, P1 implies Q1(i) and Q2, and all six properties P*1, $\cdots$, Q*3 are equivalent.

E. R. Kolchin (Paris).

Lang, Serge. Some applications of the local uniformization theorem. Amer. J. Math. **76**, 362–374 (1954).

Let K be a function field over a constant field k such that K/k is regular. Suppose that k is complete under a valuation but not necessarily algebraically closed. Let $\mathfrak{M}$ be the set of all rational places of K/k and assume that $\mathfrak{M}$ is not empty. $\mathfrak{M}$ is called the rational Riemann manifold of K/k. The set $\mathfrak{M}$ is topologized as usual [C. Chevalley, Introduction to the theory of algebraic functions of one variable, Math. Surveys, no. 6, Amer. Math. Soc., New York, 1951; these Rev. **13**, 64] such that the map $x\rightarrow x(\mathfrak{p})$ ($\mathfrak{p}\in\mathfrak{M}$) is continuous for each $x\in K$. The author proves that, in case of characteristic 0, there exist infinitely many rational places which are enough to distinguish between elements of K. The proof makes use of the local uniformization theorem of Zariski [Ann. of Math. (2) **41**, 852–896 (1940); these Rev. **2**, 124]. In general, if k is locally compact then $\mathfrak{M}$ is compact. In particular, when K/k is a regular extension of dimension 1 then $\mathfrak{M}$ is locally homeomorphic with k. Finally the author remarks on the parallel with the results on real function fields which were developed earlier [S. Lang, ibid. **57**, 378–391 (1953); these Rev. **14**, 841]. *Y. Kawada.*

Jenner, W. E. Block ideals and arithmetics of algebras. Compositio Math. **11**, 187–203 (1953).

Consider two-sided ideals of a ring $\mathfrak{O}$ with 1. Denote by $\bigcap^* b_i$ the direct intersection of ideals b_i (i.e. the intersection of ideals $\neq(1)$ relatively prime in pairs). Call a a block ideal if it is not expressible as a direct intersection of more than one ideal. Necessary and sufficient for this is that $\mathfrak{O}/a$ is directly indecomposable. An ideal a has a unique expression as a direct intersection of block ideals if the maximum condition holds for ideals containing a, or if the minimum condition holds for left (right) ideals containing a.

Take $a=\bigcap^*_{1\leqq j\leqq r} b_j$ where the b_j are block ideals, and assume the minimum condition holds for left (right) ideals of $\mathfrak{O}/a$. Let $P=(p_i)_{1\leqq i\leqq s}$ be the set of all prime ideal divisors of a and B_j the set of all p_i dividing b_j. Then there exist elements α_i ($1\leqq i\leqq s$) which are orthogonal idempotents mod a such that $\alpha_i\equiv\delta_{ik}$ (mod p_k) ($1\leqq i, k\leqq s$) and $\sum_{1\leqq i\leqq s}\alpha_i=1$ hold. Two distinct elements p_i, p_k of P belong to the same "block" if and only if a chain $p_i, \cdots, p_r, \cdots, p_k$ of elements of P exists such that for any two adjacent chain elements either $\alpha_i\mathfrak{O}\alpha_k$ non-$\subseteq a$ or $\alpha_k\mathfrak{O}\alpha_i$ non-$\subseteq a$ holds. They also belong to the same B_j if and only if a chain $p_i, \cdots, p_r, \cdots, p_k$ exists such that no two adjacent chain elements commute mod a. If all elements of P commute mod a each block contains only one p_i. The ideal n of $\mathfrak{O}$ such that n/a is the Wedderburn-Artin radical of $\mathfrak{O}/a$ satisfies $n=\bigcap^*_{p\in P} p$. If σ is the exponent of n, all $M_j=\bigcap_{p\in B_j} p$ ($1\leqq j\leqq r$) commute mod a, and $b_j=M_j^\lambda+a$ holds for any integer $\lambda\geqq\sigma$. If all elements of P commute mod (0) then all b_i commute and $a=\prod_{1\leqq i\leqq r} b_i$.

Now let $\mathfrak{o}$ be a Dedekind ring with quotient field k and A a finite-dimensional algebra over k with 1 coinciding with the $1\in k$. Then all previous results apply to integral ideals in an order $\mathfrak{O}$ of A. The block ideal components of ideals of the form $a\mathfrak{O}$ where a is a non-zero integral $\mathfrak{o}$-ideal generate a multiplicative abelian group. If Z is the center of A and p a prime ideal of $\mathfrak{o}$ then $p\mathfrak{O}=\bigcap^* B_i$ for block ideals B_i implies $p(\mathfrak{O}\cap Z)=\bigcap^* T_i$ where $T_i=B_i\cap\mathfrak{O}\cap Z$ are block ideals of $\mathfrak{O}\cap Z$. From these results the author develops then the prime ideal decomposition theorem for maximal orders.

Let A be semisimple and separable over k. If $\mathfrak{D}$ and $\mathfrak{D}^*$ are the discriminants of orders $\mathfrak{O}$ and $\mathfrak{O}^*$ resp. with $\mathfrak{O}\subseteq\mathfrak{O}^*$, then $\mathfrak{D}\subseteq\mathfrak{D}^*$ if and only if $\mathfrak{O}\subset\mathfrak{O}^*$. Let p be a prime ideal of k, $\mathfrak{D}_p$ the p-component of the discriminant of $\mathfrak{O}$, $\mathfrak{o}_p$ the ring of p-integers of k, set $\mathfrak{O}_p=\mathfrak{o}_p\mathfrak{O}$, and assume that $\mathfrak{o}/a$ is finite for every integral ideal a of k. Then the algebra $\mathfrak{O}_p/p\mathfrak{O}_p$ over $\mathfrak{o}_p/p\mathfrak{o}_p$ is semisimple if and only if $\mathfrak{D}_p=\mathfrak{o}$. If $\mathfrak{O}^*$ is maximal and $\mathfrak{O}\subseteq\mathfrak{O}^*$, the ideal generated by the set of all non-zero $\mathfrak{O}^*$-ideals which lie in $\mathfrak{O}$ is called the conductor $\mathfrak{F}$ of $\mathfrak{O}$ with respect to $\mathfrak{O}^*$. The mapping $a\rightarrow\mathfrak{O}^*a\mathfrak{O}^*$ from the set of all $\mathfrak{O}$-ideals a with $(a, \mathfrak{F})=\mathfrak{O}$ onto the set of all $\mathfrak{O}^*$-ideals a^* with $(a^*, \mathfrak{F})=\mathfrak{O}^*$ is an isomorphism with respect to sum, product, and intersection. If A is separable $\mathfrak{D}_p\subseteq\mathfrak{D}_p^*$ holds if and only if p is divisible by a prime ideal divisor in $\mathfrak{O}^*$ of $\mathfrak{F}$. *A. Jaeger* (Cincinnati, Ohio).

Kasch, Friedrich. Grundlagen einer Theorie der Frobeniuserweiterungen. Math. Ann. **127**, 453–474 (1954).

The author's notion of "Frobenius extension" generalizes Nakayama's theory of "Frobenius algebras". The author first observes that if A is a ring satisfying the minimal conditions for left and right ideals, having a unit and in which, in addition, proper left and right ideals have non-zero annihilators (right and left, respectively) (in which case the author says A is an S-ring), then the usual duality theory for vector spaces over a sfield carries over almost completely for A-modules having a basis; a regular duality between a left A-module L and a right A-module R (having bases) is defined by a bilinear mapping $(x, y)\rightarrow\langle x, y\rangle$ of $L\times R$ into A such that $\langle x, R\rangle=0$ implies $x=0$ and $\langle L, y\rangle=0$ implies $y=0$. Now let S be a ring with unit which is both left and right A-module (with bases); if $(x, y)\rightarrow\langle x, y\rangle$ is a regular duality between S (left) and S (right) such that $\langle xs, y\rangle=\langle x, sy\rangle$ for all $s\in S$, the bilinear mapping $\langle x, y\rangle$ is entirely determined by the mapping $x\rightarrow\langle x\rangle=\langle 1, x\rangle=\langle x, 1\rangle$ of S into A, which is both left and right A-linear, and whose kernel contains no ideal $\neq(0)$. Conversely, such a mapping defines a regular duality $(x, y)\rightarrow\langle x, y\rangle$ of the preceding type; when such a mapping exists, S is called a Frobenius extension of A. A group ring over A is a Frobenius extension of A; so is the tensor product (over a commutative S-ring K) of a Frobenius extension of K and of an S-ring A, K being in the center of both A and S; in addition, the author points out that a simple ring is a Frobenius extension of a Galois subring [in the sense of Nakayama, Trans. Amer. Math. Soc. **73**, 276–292 (1952); these Rev. **14**, 240]. The usual properties of ideals in a Frobenius algebra can be extended to ideals in a Frobenius extension S of A which are modules over A, with bases; conditions are given under which a quotient ring of S by such an ideal is still a Frobenius extension of A. Finally, the author applies his theory to generalize still further the abstract formulation of Maschke's theorem recently given by Gaschütz [Math. Z. **56**, 376–387 (1952); these Rev. **14**, 533]; one of the questions treated is the following one: if B is a subring of a ring A, one wants to characterize left A-modules M such that when they are direct summands of an A-module N, as B-modules, they are also direct summands of N as A-modules. The author obtains such a characterization when A is a Frobenius extension of B. *J. Dieudonné* (Evanston, Ill.).

Segre, Beniamino. La teoria delle algebre ed alcune questioni di realtà. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **13**, 157–188 (1954).

Soient K un corps, et A une algèbre non nécéssairement associative de dimension finie sur K. L'auteur remarque que l'étude de certaines propriétés de A (p. ex., la recherche d'idempotents ou d'éléments de carré nul) se traduit par la

résolution d'un système d'équations quadratiques sur K, c.à.d. par l'étude des points rationnels sur K d'une intersection d'hyperquadriques. Ainsi un raisonnement simple montre qu'une algèbre réelle ou complexe sans éléments nilpotents admet toujours des idempotents. De même, le théorème démontré par H. Hopf par voie topologique et disant que toute algèbre commutative (non nécéssairement associative) de dimension $n>2$ sur les réels admet des diviseurs de zéro [Univ. e Politecnico Torino. Rend. Sem. Mat. 11, 75–91 (1952); ces Rev. 14, 720], équivaut au fait que, pour tout $q>1$ et pour tout système linéaire de dimension q d'hyperquadriques à coéfficients réels de P_q, il existe au moins un couple de points réels de P_q conjugués par rapport à toutes les hyperquadriques du système. Ceci se démontre en considérant, dans l'espace projectif des enveloppes réelles de seconde classe, les points (réels) d'intersection de divers hyperplans avec les variétés des enveloppes décomposées en deux points réels, ou en deux points complexes conjugués, ou en un point réel double. Il résulte aussi de cette étude, que le nombre des solutions réelles d'un système de $2n$ équations bilinéaires symétriques à coéfficients réels en 2 séries de $n+1$ variables, est pair (resp. impair) si n est (resp. n'est pas) une puissance de 2. Des résultats plus précis, et améliorant ceux de Stiefel [Comment. Math. Helv. 13, 201–218 (1941); ces Rev. 3, 61], peuvent s'obtenir en considérant la courbe jacobienne d'un système linéaire de dimension 4 de quadriques de P_3. Les démonstrations de ce mémoire sont pûrement algébriques, et sont donc valables pour on corps ordonné maximal quelconque. *P. Samuel.*

Plattner, P. Anton. Die charakteristische Funktion von Hilbert für Potenzen von Hauptklassenidealen. Monatsh. Math. 58, 103–113 (1954).

Explicit expressions are obtained for the Hilbert characteristic function $H(t, A^\rho)$ of a power of a homogeneous polynomial ideal A of principal class, which has a basis of forms of the same degree τ. If r is the rank of A and if $H(t; A^\rho)=\sum h_i C(t, d-i)$ $(t\geqq T)$, it is found that h_i is a polynomial in τ of degree $r+i$, and in each of these polynomials, the coefficient of τ^j is a polynomial in ρ of degree j. *H. J. Muhly* (Iowa City, Iowa).

Aubert, Karl Egil. Some characterizations of valuation rings. Duke Math. J. 21, 517–525 (1954).

Expanding on a previous result [Math. Ann. 127, 8–14 (1954); these Rev. 15, 501] the author gives necessary and sufficient conditions for an integral domain to be (i) a valuation ring, (ii) a valuation ring with value group discrete in the order topology (called "discrete valuation ring"), (iii) a discrete, rank-one valuation ring. The conditions are in terms of the r-systems of Prüfer [cf. review referred to above for notations]. Sample theorem: An integral domain is a discrete valuation ring if and only if every s-ideal is a v-ideal (i.e., all r-systems coincide). There is also one theorem describing, in similar terms, a certain class of rings with divisors of zero that are analogous to discrete valuation rings. *D. Zelinsky* (Evanston, Ill.).

Zelinsky, Daniel. Every linear transformation is a sum of nonsingular ones. Proc. Amer. Math. Soc. 5, 627–630 (1954).

Let M and N be vector spaces of equal dimension (finite or infinite) over a division ring F, and let α be a linear mapping of M into N. It is shown that α is expressible as a sum $\beta+\gamma$, where β and γ are isomorphisms of M onto N, except when M and N are one-dimensional over GF(2), and α is the only non-zero linear mapping of M into N. Thus the ring $T(F, A)$ of all linear transformations of the vector space A, over the division ring F, is generated by its units (the non-singular linear transformations = automorphisms of the space). It is pointed out that a proof of this latter result could also be based on the reviewer's result [Amer. J. Math. 75, 358–386 (1953); these Rev. 14, 718] that $T(F, A)$ is generated by its idempotents, when the dimension of A is greater than one. *K. G. Wolfson.*

Wolfson, Kenneth G. Some remarks on ν-transitive rings and linear compactness. Proc. Amer. Math. Soc. 5, 617–619 (1954).

A ring K of linear transformations is ν-transitive in case each independent set of less than $\aleph_\nu$ vectors can be sent into arbitrarily prescribed vectors by transformations in K. A characterization of such a ring independent of its representation as linear transformations is: the socle is not a zero ring and is contained in every nonzero ideal; and any collection of less than $\aleph_\nu$ W-cosets with the finite intersection property has a nonvoid intersection. Here "W-coset" means a coset of an intersection of a finite number of maximal right annihilators in K. If a primitive ring of linear transformations is linearly compact in a topology where W-cosets are closed, then it is the ring of all linear transformations. *D. Zelinsky* (Evanston, Ill.).

Iséki, Kiyoshi. On 0-dimensional compact ring. Math. Japonicae 3, 37–40 (1953).

Let R be a zero-dimensional compact ring. Then it is proved that every neighborhood of the zero has a compact open two-sided ideal. [This is a direct consequence of Lemmas 9 and 10 of Kaplansky, Amer. J. Math. 69, 153–183 (1947); these Rev. 8, 434.] From this, it is obtained, in the usual way, that R is the inverse limit of finite rings R_α. The author then shows that: (1) an element in R is quasi-regular if and only if its projection on each R_α is quasi-regular; (2) R has a unit if and only if each R_α has a unit; (3) an element in R has an inverse if and only if its projection on each R_α has an inverse. *P. S. Mostert.*

Theory of Groups

MacKenzie, Robert E. Commutative semigroups. Duke Math. J. 21, 471–477 (1954).

This paper generalizes to commutative semigroups the E. Noether theory of decomposition of ideals of a commutative ring into the intersection of primary ideals. This is accomplished by considering a family $\mathfrak{F}$ of ideals of a commutative semigroup $\mathfrak{M}$ satisfying three conditions which are also satisfied by the Dedekind ideals of a ring. This is the same general procedure as that adopted by Lorenzen, with his r-systems of ideals, but the defining conditions are different, just as the objectives are different. As it stands, the theory cannot be deduced from the residuated lattice approach of Krull, Ward, and Dilworth. In fact, the notions of product and residual of two ideals are never used. The author operates entirely with complements of ideals, called co-ideals, and the whole paper is therefore the exact dual of what one is used to. This review will be couched in the usual terminology. The three conditions on the family $\mathfrak{F}$ are: (I) if $\mathfrak{a}$ belongs to $\mathfrak{F}$, so does its radical; (II) the intersection of any set of members of $\mathfrak{F}$ belongs to $\mathfrak{F}$; (III) if M is a subsemigroup of $\mathfrak{M}$ disjoint from a member $\mathfrak{a}$ of $\mathfrak{F}$, then $\mathfrak{F}$

contains the ideal $\mathfrak{a}_M$ consisting of all $x \in \mathfrak{M}$ such that $xm \in \mathfrak{a}$ for some $m \in M$ ($\mathfrak{a}_M$ is called a segregated component of $\mathfrak{a}$). Several familiar theorems are shown, including the uniqueness theorem for finite intersections of irredundant primary ideals with distinct radicals, and Krull's theorem on the existence of a minimal isolated primary component of an ideal. If $\mathcal{P}$ is a set of prime ideals of $\mathfrak{F}$ containing $\mathfrak{a}$, then the isolated component $\mathfrak{a}:\mathcal{P}$ is defined to be the intersection of all primary ideals in $\mathfrak{F}$ containing $\mathfrak{a}$ and contained in some member of $\mathcal{P}$. If $\mathfrak{a}$ is the intersection of a finite number of primary ideals, then every segregated component of $\mathfrak{a}$ is also an isolated component, but not in general conversely. It is shown that every ideal in $\mathfrak{F}$ is the intersection of a finite number of primary ideals in $\mathfrak{F}$ if every ideal $\mathfrak{a}$ in $\mathfrak{F}$ satisfies both of the following conditions: (a) For each segregated component $\mathfrak{a}_M$ of $\mathfrak{a}$ there exists an ideal $\mathfrak{b} \neq \mathfrak{a}$ in $\mathfrak{F}$ such that $\mathfrak{a}=\mathfrak{a}_M \cap \mathfrak{b}$, and such that $\mathfrak{a}=\mathfrak{a}_M \cap \mathfrak{c}$ implies $\mathfrak{b} \supseteq \mathfrak{c}$. (b) If $\mathfrak{a}:\mathcal{P}_1 \supseteq \mathfrak{a}:\mathcal{P}_2 \supseteq \cdots$ is a descending chain of isolated components of $\mathfrak{a}$, then all but a finite number of its terms are equal. *A. H. Clifford* (Baltimore, Md.).

Parker, E. T. On multiplicative semigroups of residue classes. Proc. Amer. Math. Soc. **5**, 612–616 (1954).

The "basic" semigroup $\sigma(p^k, m)$ is defined to be the commutative semigroup generated by x and y where $x^r y^s = x^u y^v$ if and only if either $0 \leqq s = v < m$ and $r \equiv u(p^k)$ or $s \geqq m$ and $v \geqq m$. Then it is proved that a semigroup can be embedded in a multiplicative semigroup of residue classes if and only if it is embeddable in a direct product of finitely many basic semigroups. An example is given to show that this cannot be done for every finite commutative semigroup.

H. A. Thurston (Bristol).

Ivan, Ján. On the direct product of semigroups. Mat.-Fyz. Časopis. Slovensk. Akad. Vied **3**, 57–66 (1953). (Slovak. Russian summary)

Let S_1 and S_2 be semigroups without zero, and $S=S_1 \times S_2$ their direct product. Then S contains a minimal left (right, two-sided) ideal if and only if S_1 and S_2 both contain minimal left (right, two-sided) ideals. S is simple if and only if S_1 and S_2 are both simple. S is a class sum of groups if [and only if] the same is true of S_1 and S_2. *A. H. Clifford.*

Liber, A. E. On the theory of generalized groups. Doklady Akad. Nauk SSSR (N.S.) **97**, 25–28 (1954). (Russian)

A "generalized group" was defined by V. V. Vagner [Mat. Sbornik N.S. **32**(74), 545–632 (1953); these Rev. **15**, 501] to be a semigroup (associative system) G satisfying the following two conditions: (1) each element g of G possesses a "generalized inverse" $\bar{g}$ in G: $g\bar{g}g=g$ and $\bar{g}g\bar{g}=\bar{g}$; (2) any two idempotent elements of G commute. Vagner showed that $\bar{g}$ in (1) is unique. The present author shows conversely, that if (1) holds with unique $\bar{g}$, then (2) holds. G is called "simplest" if, in addition, $gg^{-1}=g^{-1}g$ for all g in G. It is shown that G is simplest if and only if the set I of all idempotent elements of G is contained in the center of G. A simplest generalized group is a class sum of groups with commuting identity elements, the structure of which was found by the reviewer [Ann. of Math. (2) **42**, 1037–1049 (1941); these Rev. **3**, 199]. Suppose now that G is a generalized group such that I contains a zero element i_0. Let G_0 be the set of all elements g of G such that $g^{-1}g=i_0$. Then $gg^{-1}=i_0$ also, and G_0 is a subgroup and an ideal of G. [In other words, G_0 is the group of zeroid elements of G.] Suppose that I contains an identity element e. Then e is also an identity element of G. The set G_e of all g in G such that $g^{-1}g=e$ is a subsemigroup of G, not in general a group, since we may have $g^{-1}g=e$ but $gg^{-1} \neq e$. If, however, I is finite, then G_e is a group. If I has just two elements, or just three elements one of which is an identity element, then G is simplest. *A. H. Clifford* (Baltimore, Md.).

Szele, T. On the basic subgroups of abelian p-groups. Acta Math. Acad. Sci. Hungar. **5**, 129–141 (1954). (Russian summary)

After reviewing Kulikoff's [Mat. Sbornik N.S. **16**(58), 129–162 (1945); these Rev. **8**, 252] concept of a basic subgroup B of a primary abelian group G, the author proves that B is a homomorphic image of G. The proof may be paraphrased as follows. B is a direct sum of cyclic groups. If T is the torsion subgroup of their complete direct sum, there is a homomorphism of G into T. This is to be followed by the homomorphism of T onto B obtained by sending the $2n$th component into the nth. From this the author deduces that if a direct sum of cyclic groups is a homomorphic image of G, it is also a homomorphic image of B.

I. Kaplansky (Chicago, Ill.).

Fuchs, L. On a property of basic subgroups. Acta Math. Acad. Sci. Hungar. **5**, 143–144 (1954). (Russian summary)

The second theorem in the paper of the preceding review is proved more simply. *I. Kaplansky* (Chicago, Ill.).

Azleckiĭ, S. P. On normal series of Sylow classes of minimal systems of a finite group. Mat. Sbornik N.S. **34**(76), 269–278 (1954). (Russian)

This paper continues the author's study of finite groups and their normal subgroups generated by classes of Sylow subgroups [see Mat. Sbornik N.S. **28**(70), 461–466; **29**(71), 581–586 (1951); **31**(73), 359–366 (1952); these Rev. **12**, 799; **13**, 721; **14**, 532]. Let p_i ($i=1, \cdots, k$) be the distinct primes which divide the order of G; let $\langle P_i \rangle$ be the normal subgroup generated by the class of Sylow subgroups corresponding to the prime p_i. For minimal r, a collection $\langle P_1 \rangle, \langle P_2 \rangle, \cdots, \langle P_r \rangle$ which generates G is called a minimal system of Sylow classes for G; r is called the Sylow rank of G. A chain T_i ($i=0, \cdots, r$) of normal subgroups of G (T_0=identity, $T_r=G$) such that T_{i+1} is the group generated by T_i and $\langle P_{j_i} \rangle$, where $\{j_i\}$ is a permutation of $\{1, \cdots, r\}$ is called a normal series for the given minimal system of Sylow classes. Theorems: 1. If the Sylow rank of G is k, then the set of indices of a normal series is independent of the permutation $\{j_i\}$ if and only if G is a special group. 2. G is a special group if and only if the indices in a normal series are prime powers. 5. If the set of indices in a normal series is independent of the permutation $\{j_i\}$, it is necessary that G be the direct product of r Sylow classes. 3. If G has a unique minimal system, this condition is also sufficient. (Note that P_t and P_u are considered different if t, u are different, even if $\langle P_t \rangle = \langle P_u \rangle$.) 6. If G is not solvable, and if r is equal to the length of a principal series, G must have more than one minimal system. 8(9). If r is equal to the length of a composition (principal) series, then G is the direct product of the $\langle P_i \rangle$ as in 5, and each $\langle P_i \rangle$ is simple (an elementary group). 7. If (in 8) moreover G has just one minimal system, G is the direct product of groups of prime order.

J. L. Brenner (Aberdeen, Md.).

Čunihin, S. A. On the decomposition of Π-separable groups into a product of subgroups. Doklady Akad. Nauk SSSR (N.S.) 95, 725–727 (1954). (Russian)

[For terminology, see same Doklady (N.S.) **86**, 27–30 (1952); these Rev. **14**, 350.] Let m be the maximal Π-Sylow divisor of the order g of the Π-separable group $\mathfrak{G}$. To each factorization $m=m_1m_2$ with $(m_1, m_2)=1$ corresponds a factorization $\mathfrak{G}=\mathfrak{M}_1\mathfrak{M}_2$, where $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are subgroups of $\mathfrak{G}$ such that m_j is the maximal Π-Sylow divisor of the order of $\mathfrak{M}_j$ for $j=1, 2$. Repeated application of this result yields a factorization of $\mathfrak{G}$ as a product of as many subgroups as there are prime-power factors in the decomposition of m.

R. A. Good (College Park, Md.).

Plotkin, B. I. Lattice isomorphisms of soluble R-groups. Doklady Akad. Nauk SSSR (N.S.) 95, 1141–1144 (1954). (Russian)

The author studies properties of a group $\mathfrak{G}$ preserved under a lattice isomorphism ϕ onto the group $\mathfrak{G}^\phi$. If $\mathfrak{G}$ has an ascending invariant solvable series such that $\mathfrak{G}/\mathfrak{A}_\alpha$ is an R-group for each member $\mathfrak{A}_\alpha$ of the series, then the image of this series under ϕ is a series in $\mathfrak{G}^\phi$ of the same type. If $\mathfrak{G}$ is an R^*-group and has an ascending invariant solvable series, then (1) $\mathfrak{G}^\phi$ has the same properties, (2) the image under ϕ of an invariant isolated subgroup of $\mathfrak{G}$ is an invariant isolated subgroup of $\mathfrak{G}^\phi$, (3) the image under ϕ of the isolator of the commutant of $\mathfrak{G}$ is the isolator of the commutant of $\mathfrak{G}^\phi$. *R. A. Good* (College Park, Md.).

Berlinkov, M. L. Groups having a compact lattice of subgroups. Mat. Sbornik N.S. 34(76), 473–498 (1954). (Russian)

A convergent sequence of sets (A_n) is one for which the union $\bigcup\{\bigcap_{i=n}^{\infty} A_i\}$ and the intersection $\bigcap\{\bigcup_{i=n}^{\infty} A_i\}$ are equal. A group with compact lattice (c.l.) is one in which every sequence of subgroups has a convergent subsequence. A finite or quasi-cyclic group has c.l.; every group with c.l. is periodic, but not conversely. An infinite p-group with c.l. is either quasi-cyclic or does not satisfy the descending chain condition (d.c.c.). A locally finite group which is the direct product of its Sylow p-subgroups has c.l. if and only if the Sylow p-subgroups are either finite or quasi-cyclic. An example is given of a locally finite group with c.l. not a direct product as above. If a group has c.l. and satisfies d.c.c., then G is a direct product the factors of which are quasi-cyclic Sylow subgroups, and a finite group of order prime to the orders of these quasi-cyclic factors and conversely. If G has c.l., then every complete abelian subgroup $(\neq I)$ is a Sylow Π-subgroup contained in the center of G. Connections are given in other theorems with the concepts of local normality and resolvability, e.g., a locally normal group has c.l. if and only if every infinite p-subgroup is quasi-cyclic; a solvable group with c.l. is locally normal. In the last section of the paper, a sequence is called reduced if every subsequence has the same lower and the same upper limit. In a countable group, every sequence of subgroups contains a reduced subsequence, but this is not true for groups with the power of the continuum which have an irreducible system of generators. *J. L. Brenner.*

Neumann, B. H. An essay on free products of groups with amalgamations. Philos. Trans. Roy. Soc. London. Ser. A. 246, 503–554 (1954).

Extract from the introduction: ". . . the greater part of this essay is a translation of the 'Anhang' to appear with Kurosch's Gruppentheorie [Akademie-Verlag, Berlin, 1953] . . . I have used this opportunity to revise and supplement . . .". In my review of Kurosch's book [these Rev. **15**, 681] I gave a brief summary, as was appropriate for the appendix of a long book. I now give a more detailed review.

Pertinent references are the following papers by various combinations of G. Higman, B. H. Neumann and H. Neumann. All except the last appeared in the J. London Math. Soc. (1) **12**, 120–127 (1937). (2) **18**, 4–11 (1943); these Rev. **5**, 58. (3) **24**, 247–254 (1949); these Rev. **11**, 322. (4) **25**, 247–248 (1950); these Rev. **12**, 390. (5) **26**, 59–61 (1951); these Rev. **12**, 390. (6) **26**, 61–64 (1951); these Rev. **12**, 390. (7) **28**, 351–353 (1953); these Rev. **15**, 8. (8) Proc. London Math. Soc. (3) **1**, 284–290 (1951); these Rev. **13**, 430.

The central theme is the generalized free product (or free product with amalgamated subgroups). In Ch. I the generalized free product is defined and certain special cases are studied. Ch. II presents criteria for the existence of the generalized free product. A final section is devoted to the abelian case. Ch. III treats the special case of one amalgamated subgroup; here by a theorem of Schreier the generalized free product always exists. A rapid corollary is the author's theorem (2) that any group can be embedded in a group in which every element is an nth power for every n. The final section takes up the generalized direct product, which is defined in a fashion analogous to the generalized free product.

In Ch. IV various applications are taken up. If A and B are isomorphic subgroups of G, then G can be enlarged to a group in which the isomorphism becomes inner (3). An alternative unpublished proof due to Philip Hall is presented; it uses permutation techniques rather than free products. Any group can be embedded in a larger group in which every two elements of equal order are conjugate (3). Every countable group can be embedded in a group with two generators (3). Combining this with the author's theorem (1) that there are continuum many two-generator groups, one answers in the negative Malcev's question as to whether there is a "universal" countable group, i.e. one containing an isomorphic copy of every countable group. Ch. V is entitled "Three remarkable groups of Graham Higman's". Two problems proposed by H. Hopf are answered in the negative by suitable examples. In (4) the author gave an example of a finitely generated group isomorphic to a proper factor group. Higman (5) gave an example of this kind which was even finitely related. The same construction yields two finitely generated non-isomorphic groups each of which is a homomorphic image of the other (7). Next there are a number of theorems concerning subgroups of finite rank of a free group. A typical one (5) is that no subgroup of finite rank of a free group can be mapped onto a proper supergroup by an automorphism of the free group. The second group of Higman's (6) is a finitely generated infinite simple group, and the third (8) is a non-free group such that every countable subgroup is free. *I. Kaplansky.*

Chehata, C. G. Simultaneous extension of partial endomorphisms of groups. Proc. Glasgow Math. Assoc. 2, 37–46 (1954).

Necessary and sufficient conditions are found for a well ordered collection of partial endomorphisms of a group G to be extended to a collection of endomorphisms of an extension G^* of G. In particular, this is always possible if G is abelian, and, in this case, G^* may be chosen abelian. The

present paper extends the work of B. H. Neumann and Hanna Neumann [Proc. London Math. Soc. (3) **2**, 337–348 (1952); these Rev. **14**, 351]. *F. Haimo.*

Pic, Gh. Sur le quasi-centre d'un groupe. Acad. Repub. Pop. Romăne. Stud. Cerc. Mat. **4**, 7–21 (1953). (Romanian. Russian and French summaries)

The quasi-center of a group G is defined as the subgroup generated by all cyclic subgroups of G that are permutable with all subgroups of G. The quasi-center therefore contains the center and Baer's nucleus of G. The author establishes the following properties of the quasi-center: It is a characteristic and nilpotent subgroup of G. If it is non-abelian, then it contains no elements of infinite order. This follows from the facts that the elements of infinite order form an abelian subgroup, and that every element of infinite order is permutable with every element of finite order. If s and t are generators of two cyclic subgroups of the quasi-center whose orders are powers of the same prime number p, then the commutator of s and t is a power product of s and t with exponents divisible by p. *K. A. Hirsch.*

Loonstra, F. Sur les extensions du groupe additif des entiers rationnels par le même groupe. Nederl. Akad. Wetensch. Proc. Ser. A. **57** = Indagationes Math. **16**, 263–272 (1954).

It is proved that the extension of the additive group of integers by the same group is equivalent to one of the following groups G_1 and G_2: G_1 is the commutative group which consists of all pairs $(m; n)$ of integers m and n and has the composition rule $(m; n)+(m'; n')=(m+m'; n+n')$; G_2 is the non-commutative group which consists of all pairs $(m; n)$ of integers m and n and has the composition rule

$$(m; n)+(2m'; n')=(m+2m'; n+n'),$$
$$(m; n)+(2m'+1; n')=(m+2m'+1, -n+n'+1).$$

H. Bergström (Göteborg).

Farahat, H. On the representations of the symmetric group. Proc. London Math. Soc. (3) **4**, 303–316 (1954).

In this subject progress has been rapid lately and it is unavoidable that the same results are obtained independently by different authors. This happens here in the case of the enumeration of the modular representations in a given p-block [cf. Osima, Canadian J. Math. **5**, 336–343 (1953); these Rev. **15**, 100; Nagao, ibid. **5**, 356–363 (1953); these Rev. **14**, 1061; Frame and Robinson, ibid. **6**, 125–127 (1954); these Rev. **15**, 682]. Farahat's proof is an extension of the independence argument used originally by Chung [ibid. **3**, 309–327 (1951); these Rev. **13**, 106]. *G. de B. Robinson* (Toronto, Ont.).

Kertész, A., and Szele, T. On the existence of non-discrete topologies in infinite abelian groups. Publ. Math. Debrecen **3** (1953), 187–189 (1954).

The following theorem is proved: Every infinite abelian group admits a non-discrete topology (making it a topological group) satisfying the first axiom of countability. An abelian group admits a non-discrete subgroup topology (i.e., there exists a neighborhood basis at the group identity consisting of subgroups only) if and only if it does not satisfy the minimum condition for subgroups. The proof utilizes a theorem by Prüfer [Math. Z. **17**, 35–61 (1923)] and Kuroš [Math. Ann. **106**, 107–113 (1932)] and proceeds as follows: If the group G contains an element of infinite order, any non-discrete subgroup topology of the integers will do. Otherwise, if G does not satisfy the minimum condition, the above Prüfer-Kuroš theorem implies that G contains an infinite subgroup which is the direct sum of an infinite number of cyclic groups, and we again obtain the desired subgroup topology. In the presence of the minimum condition, the Prüfer-Kuroš theorem implies that G contains a subgroup of type p^∞ [see Prüfer's paper for the definition of p^∞] which, qua subgroup of the circle, has a nondiscrete metrizable topology. The converse of the last part is proved in a straightforward manner and is valid also for noncommutative groups. *G. K. Kalisch* (Minneapolis, Minn.).

Harish-Chandra. Representations of semisimple Lie groups. III. Trans. Amer. Math. Soc. **76**, 234–253 (1954).

This is a continuation of parts I and II [same Trans. **75**, 185–243 (1953); **76**, 26–65 (1954); these Rev. **15**, 100, 398]. The same terminology is employed. Let π be a quasi-simple representation of a connected semisimple Lie group G on a Hilbert space H. One of the main objects of this paper is to study the character of π. Indeed, if $f(g)$ is indefinitely differentiable and of compact support on G, then the operator $\int f(g)\pi(g)dg$ is shown to have a trace $T(f)$ which is a distribution in the sense of L. Schwartz, called the character of π. For the complex unimodular group these characters have previously been studied by Gelfand and Naimark [see, e.g., Trudy Mat. Inst. Steklov. **36** (1950); these Rev. **13**, 722] who obtained explicit formulas. In the present paper it is proved that two quasi-simple representations π_1 and π_2 of G are infinitesimally equivalent in the sense of the author (see parts I and II) if and only if they have the same characters. If π_1 and π_2 are irreducible unitary this implies equivalence in the usual sense.

In the last section the author derives an explicit formula for the elementary spherical functions in terms of exponential polynomials provided G is complex semisimple. This formula was first obtained by Gelfand and Naimark (see, for example, loc. cit.) for the classical complex groups. It is known that no such simple formula exists for arbitrary semisimple Lie groups and the problem of an explicit expression for the elementary spherical functions is unsolved in general and probably difficult. *F. I. Mautner.*

Harish-Chandra. On the Plancherel formula for the right-invariant functions on a semisimple Lie group. Proc. Nat. Acad. Sci. U. S. A. **40**, 200–204 (1954).

Let G be a semisimple Lie group. Without loss of generality for the problem of the present paper one can assume that the center of G is finite. Let K be a maximal compact subgroup of G, and $L_2(G/K)$ the Hilbert space of square integrable functions on the coset space G/K relative to Haar measure. The author studies the decomposition of the natural unitary representation of G in $L_2(G/K)$. This problem has previously been studied by the reviewer [same Proc. **37**, 529–533 (1951); these Rev. **13**, 434]. The present author outlines a proof that the Plancherel formula for $L_2(G/K)$ is essentially equivalent to the simultaneous eigenfunction expansion under a finite number of basic (in general partial) differential operators. [In this connection see also Gelfand, Doklady Akad. Nauk SSSR (N.S.) **70**, 5–8 (1950); these Rev. **11**, 498.] *F. I. Mautner* (Princeton, N. J.).

Tits, J. Le plan projectif des octaves et les groupes exceptionnels E_6 et E_7. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **40**, 29–40 (1954).

The Cayley projective plane is a non-Desarguesian projective geometry. It is a variety of dimension 16 and its lines

are 8-spheres. Except for a line at infinity it can be coordinatized by pairs of Cayley numbers. Chevalley, in his review of the author's previous paper [same Bull. (5) **39**, 309–329 (1953); these Rev. **14**, 947] noted a gap in the proof that the collineation group is a real form of the exceptional Lie group E_6. The author fills this gap and continues with the study of hermitian polarities which he relates to the exceptional Jordan algebra. Finally he gives further details of his promised interpretation of the exceptional group E_7. Using homogeneous coordinates in a five-dimensional projective space over a (skew-)field with a conjugate operator, he constructs a "quadric" which carries a large family of projective planes. Although there is no projective geometry of dimension five over the non-associative Cayley numbers, the author is able to construct an analogous 33-dimensional variety which carries a 27-dimensional family of Cayley projective planes. He states that the automorphism group of this configuration is the real form $E_{(7-25)}$ of the exceptional group E_7. *A. M. Gleason* (Cambridge, Mass.).

Bott, Raoul. On torsion in Lie groups. Proc. Nat. Acad. Sci. U. S. A. **40**, 586–588 (1954).

Let G be a semisimple compact connected and simply connected Lie group. It is shown how the "diagram" of G completely determines the integral homology groups of $\Omega(G)$ and G/T, where $\Omega(G)$ is the space of loops on G and T is a maximal torus of G. It follows that $\Omega(G)$ and G/T have no torsion and their Betti numbers vanish in odd dimensions. Moreover, if G is simple, then $\pi_3(G)$ is free cyclic. Proofs are sketched. They make use of the Morse critical-point theory and do not need the classification of Lie groups.

S. Chern (Chicago, Ill.).

Dieudonné, Jean. Groupes de Lie et hyperalgèbres de Lie sur un corps de caractéristique $p>0$. Comment. Math. Helv. **28**, 87–118 (1954).

In the theory of algebraic groups over fields of characteristic p, the mechanism of Lie algebras breaks down in that different irreducible algebraic groups may have the same Lie algebra. This paper proposes a substitute for the Lie algebra, the so-called hyper Lie algebra, which is actually a suitable amplification of the enveloping algebra of the ordinary Lie algebra. As in the classical theory of Lie groups, an algebraic group gives rise to a formal Lie group which is defined by a system of formal power series. Following Bochner [Ann. of Math. (2) **47**, 192–201 (1946); these Rev. **7**, 413] a formal Lie group of dimension n over a field K is defined by giving a system $\varphi(x, y)$ of n formal integral power series $\varphi_i(x, y)=\varphi_i(x_1, \cdots, x_n;\ y_1, \cdots, y_n)$ in $2n$ independent variables, with coefficients in K, such that $\varphi(x, 0)=x$, $\varphi(0, y)=y$, and, for a third set $z=(z_1, \cdots, z_n)$ of independent variables, $\varphi(\varphi(x, y), z)=\varphi(x, \varphi(y, z))$. A homomorphism of such a formal Lie group $\varphi(x, y)$ of dimension n into another formal Lie group $\varphi'(x', y')$ of dimension n' is defined as a system $f(x)$ of n' formal integral power series $f_j(x)$ such that $f_j(0)=0$, and $f(\varphi(x, y))=\varphi'(f(x), f(y))$.

Let $\mathfrak{o}=\mathfrak{o}_0$ denote the ring of all formal integral power series in the variables $x_1, \cdots, x_n$, with coefficients in the field K of characteristic p. For every non-negative integer r, let $\mathfrak{o}_r$ denote the subring of $\mathfrak{o}$ consisting of the power series in $x_1^{p^r}, \cdots, x_n^{p^r}$. A K-linear transformation D of $\mathfrak{o}$ is called a semiderivation of height r if $D(\mathfrak{o}_r)\subset\mathfrak{o}_r$ and $D(fg)=fD(g)+gD(f)$, for all $f \in \mathfrak{o}_r$ and all $g \in \mathfrak{o}$. The left translation L_y on $\mathfrak{o}$ is defined by setting $L_y(f(x))=f(\varphi(y, x))$, for every $f(x) \in \mathfrak{o}$. A semiderivation D of $\mathfrak{o}$ is applicable term by term to a power series and hence can be extended in the natural way to a K-linear transformation in the ring of integral formal power series in x and y (which annihilates the power series that do not contain the x_i). With this extension being understood, a semiderivation of $\mathfrak{o}$ is said to be left-invariant if it commutes with L_y. The left-invariant semiderivations of height r of $\mathfrak{o}$ constitute a p-Lie algebra $\mathfrak{g}_r$ over K. A semiderivation of height r is said to be special if it annihilates $\mathfrak{o}_r$. Thus, a special semiderivation of height r is simply an $\mathfrak{o}_r$-linear transformation of $\mathfrak{o}$ which annihilates $\mathfrak{o}_r$. The special semiderivations belonging to $\mathfrak{g}_r$ constitute an ideal $\mathfrak{s}_r$ in $\mathfrak{g}_r$ and at the same time an associative algebra over K. Furthermore, $\mathfrak{g}_r\subset\mathfrak{s}_{r+1}$. In particular, this implies that the union $\mathfrak{G}$ of all the $\mathfrak{g}_r$ is an associative algebra over K. This is the hyper Lie algebra associated with the formal Lie group $G=\varphi(x, y)$. The ordinary Lie algebra of G coincides with $\mathfrak{g}_0$.

The results of this paper indicate that the connection between a formal Lie group G and its hyper Lie algebra $\mathfrak{G}$ may be as close as that between a Lie group and its Lie algebra in the classical theory. Thus, for instance, it is proved that G is abelian if and only if $\mathfrak{G}$ is commutative. Furthermore, it is shown that with every homomorphism of a formal Lie group G into another formal Lie group G' there is associated a derived homomorphism of $\mathfrak{G}$ into $\mathfrak{G}'$ which maps each $\mathfrak{g}_r$ into $\mathfrak{g}_r'$ and each $\mathfrak{s}_r$ into $\mathfrak{s}_r'$. Moreover, this derived homomorphism of $\mathfrak{G}$ into $\mathfrak{G}'$ determines the homomorphism of G into G' completely. On the other hand, it is shown by means of an example that these properties of derived homomorphisms, even together with a further property relating to the pth power map, do not suffice to characterize derived homomorphisms.

In the case where the field K is perfect, the author obtains a very strong normalization theorem for homomorphisms u of a formal Lie group G into a formal Lie group G'. This theorem says that, after a suitable change of variables in G and G', u becomes such that, for each i, $u_i(x)=(x_{i'}')^{p^{i'}}$, with $i'\leqq i$; or $u_i(x)=0$, and then $u_j(x)=0$ for all $j\geqq i$.

It would lead too far afield to describe here the author's results on the structure of $\mathfrak{G}$, and of the algebra of semiderivations generally. They amount to a very detailed and precise analysis, but still fall short of giving a complete characterization of those hyper Lie algebras which actually rise from a formal Lie group. This problem appears to be of considerable difficulty, and its solution would be the most important finishing touch to the theory. *G. Hochschild.*

NUMBER THEORY

Brown, Alan L. Multiperfect numbers. Scripta Math. **20**, 103–106 (1954).

Lists of hitherto unpublished multiply perfect numbers n such that $\sigma(n)=kn$ are given for $k=5, 6, 7, 8$. The number of numbers given for each k are 6, 35, 66, 3, respectively, a total of 110, of which 79 are due to the author and 31 were discovered by P. Poulet but never published. The numbers are given in factored form and most of them are exceedingly large. [For another list of 63 multiperfects of which 25 are reproduced here see B. Franqui and M. Garcia, Amer. Math. Monthly **60**, 459–462 (1953); these Rev. **15**, 101.]

D. H. Lehmer (Berkeley, Calif.).

Kurepa, G. Über die Binomialkoeffizienten. Bull. Soc. Math. Phys. Serbie 5, no. 3–4, 33–44 (1953). (Serbo-Croatian summary)

n, r are integers, $r\geqq 0$, $n\geqq r$, $[x]$ denotes the greatest integer not exceeding x, and

$$\binom{n}{r}=C_{nr}=\frac{n!}{r!(n-r)!} \text{ if } r\geqq 1,\ C_{n0}=1 \text{ for all } n\geqq 0,\ \left[\frac{n}{2}\right]=n'.$$

The following are the chief results established. If p is prime, $n\geqq 1$ and $r<p^n$, $\binom{p^n}{r}$ is divisible by p; $\binom{p^n}{\alpha p^{n-1}}$ $(\alpha=1, 2, \cdots, p-1)$ is not divisible by p^2; $\binom{p^n-1}{r}$ is not divisible by p. If $p^k\leqq n<2(p^k-1)$ then $C_{nn'}$ is divisible by p (k is an integer $\geqq 1$). If $n>1$ then $\binom{n}{n'}$ has a prime factor q such that q^2 does not divide $\binom{n}{n'}$ and $\binom{n}{n'-1}$ has a prime factor r such that r^2 does not divide $\binom{n}{n'-1}$.

Corresponding to any integer $r\geqq 0$ there exists an integer r_0 such that, for all $n>r_0$, $\binom{n}{n'-r}$ has at least one prime factor s such that s^2 does not divide $\binom{n}{n'-r}$. The solutions of $\binom{n}{2}=k^2$ are given by $n=(x_r+1)/2$, $k=y_r$ $(r=1, 2, \cdots)$, where x_r and y_r are determined by $(3+\sqrt{8})^r=x_r+y_r\sqrt{8}$ $(r=1, 2, \cdots)$. Either $\binom{n}{2}$ and $\binom{(2n-1)^2}{2}$ are both perfect squares or neither of them is a perfect square. The equations $\binom{2n}{2}=k^2$ and $\binom{a}{2}=2\binom{b}{2}$ are also considered.

G. A. Dirac (London).

Turán, Pál. On a problem in the history of Chinese mathematics. Mat. Lapok 5, 1–6 (1954). (Hungarian. Russian and English summaries)

A proof of the identity

$$\sum_{j=0}^{k}\binom{k}{j}^2\binom{n+2k-j}{2k}=\binom{n+k}{k}^2$$

is given which occurred without proof in a book of the Chinese mathematician Le-Jen Shoo from 1867. Some properties of the Legendre polynomials are used.

Author's summary.

Domar, Yngve. On the Diophantine equation

$$|Ax^n-By^n|=1,\quad n\geqq 5.$$

Math. Scand. 2, 29–32 (1954).

Modifications of a method of Siegel [Math. Ann. 114, 57–68 (1937)] are used to prove the following two theorems. The Diophantine equation $|Ax^n-By^n|=1$ where A and B are positive integers and $n\geqq 5$ has at most two solutions in positive integers x and y. The equation $|x^n-My^n|=1$ with M positive and $n\geqq 5$ has at most one solution in positive integers, except possibly when $M=2$, and if $n=5$ or 6 when $M=2^n\pm 1$. The latter is, for $n=5$, an improvement of a result of P. Häggmark [Thesis, Univ. of Uppsala, 1952; these Rev. 14, 354].

I. Niven (Eugene, Ore.).

Obláth, R. Über die Gleichung $x^m+1=y^n$. Ann. Polon. Math. 1, 73–76 (1954).

Observations on the Diophantine equation $x^m+1=y^n$ under the restriction $|x-y|=1$.

I. Niven.

Mills, W. H. A method for solving certain Diophantine equations. Proc. Amer. Math. Soc. 5, 473–475 (1954).

The Diophantine equation $x^2+xyz+\epsilon y^2+ax+by+c=0$, where $\epsilon=\pm 1$, a, b, c are given integers and x, y, z are to be determined can be treated by methods evolved independently by the author [Pacific J. Math. 3, 209–220 (1953); these Rev. 14, 950] and E. S. Barnes [J. London Math. Soc. 28, 242–244 (1953); these Rev. 14, 725] but going back in essence to Hurwitz [Math. Werke, Bd 2, Birkhäuser, Basel, 1933, pp. 410–421]. If either of the polynomials x^2+ax+c, ϵy^2+by+c is factorizable over the rational field then solutions exist for all integers z; but otherwise all solutions belong to a finite number of "chains" (defined by recurrence relations); and, in particular, the equation is soluble for only a finite number of given values of z.

J. W. S. Cassels (Cambridge, England).

Vandiver, H. S. Examination of methods of attack on the second case of Fermat's last theorem. Proc. Nat. Acad. Sci. U. S. A. 40, 732–735 (1954).

In a previous paper [same Proc. 40, 25–33 (1954); these Rev. 15, 778], D. and E. Lehmer and the author proved that the equation $x^n+y^n=z^n$ had no nonzero solutions in rational integers for $2<n<2003$. In this paper, the upper limit is raised to 2521, combining as before the analytic-arithmetic criteria of Kummer and Vandiver with the computational celerity of the SWAC under the supervision of J. Selfridge. A discussion is given of the procedure to follow in the case the criteria so far employed fail.

R. Bellman (Santa Monica, Calif.).

Butler, M. C. R. On the reducibility of polynomials over a finite field. Quart. J. Math., Oxford Ser. (2) 5, 102–107 (1954).

Let $f(x)$ denote a polynomial of degree m with coefficients in $GF(p^n)$ and suppose that

$$f(x)=f_1^{k_1}(x)\cdots f_r^{k_r}(x),$$

where the $f_i(x)$ are irreducible over $GF(p^n)$. Put

$$x^{ip^n}\equiv\sum_{j=0}^{m-1}A_{ij}x^j\quad(\text{mod } f(x)),$$

where $A_{ij}\in GF(p^n)$. Then if s denotes the rank of the matrix $\|A_{ij}-\delta_{ij}\|$ $(i=1, \cdots, m-1; j=0, 1, \cdots, m-1)$, it is proved that $r=m-s$.

L. Carlitz (Durham, N. C.).

Rédei, Ladislaus. Über das Kreisteilungspolynom. Acta Math. Acad. Sci. Hungar. 5, 27–28 (1954). (Russian summary)

Let $F_n(x)$ be the nth cyclotomic polynomial. Then in the ring of polynomials with integer coefficients the ideal $(F_n(x))$ is generated by the polynomials $F_p(x^{n/p})=(x^n-1)/(x^{n/p}-1)$, where p runs through the prime divisors of n. This theorem was stated, with an incorrect proof, in a previous paper [same Acta 1, 197–207 (1950); these Rev. 13, 623]. The first correct proof was given by the reviewer [Nederl. Akad. Wetensch. Proc. Ser. A. 56, 370–377 (1953); these Rev. 15, 503]. The author now gives a new proof, which is short and elegant.

N. G. de Bruijn (Amsterdam).

Whiteman, A. L. The sixteenth power residue character of 2. Canadian J. Math. **6**, 364–373 (1954).

The theory of the division of the circle is used to prove known theorems of Gauss, Reuschle and Cunningham about 8th and 16th power residue characters of the number 2 modulo odd primes. *H. Bergström* (Göteborg).

Cohen, Eckford. A finite analogue of the Goldbach problem. Proc. Amer. Math. Soc. **5**, 478–483 (1954).

Let R_m denote the ring of residue classes modulo an integer $m>1$. The author considers the Goldbach problem in this ring and proves that every element in R_m is expressible as a sum of $G(m)$ primes in R_m if and only if m has at least two distinct prime factors and that the minimum value of $G(m)$ is 2, if m is odd, 3 if m is even and has at least two distinct odd prime factors or if m is twice an odd prime power, 4 if m is of the form $m=2^\mu p^\lambda$, $\lambda\geqq 1$, $\mu>1$, p odd. For the proof he uses one of his earlier results about the number of solutions of linear congruences. *H. Bergström.*

Carlitz, L. Congruence properties of special elliptic functions. Monatsh. Math. **58**, 77–90 (1954).

Let $\operatorname{sn} x=\operatorname{sn}(x,u)$ denote the Jacobi elliptic function, where $u=k^2$ in the standard notation. Then

$$\operatorname{sn} x=\sum_{m=0}^{\infty} A_{2m+1}(u)\frac{x^{2m+1}}{(2m+1)!}\quad (A_1(u)=1),$$

where the $A_{2m+1}(u)$ are polynomials in u with integral coefficients. Put

$$x/\operatorname{sn} x=\sum_{m=0}^{\infty}\beta_{2m}(u)x^{2m}/(2m)!\quad (\beta_0(u)=1),$$

so that the $\beta_{2m}(u)$ are polynomials in u with rational coefficients. In an earlier paper [Math. Ann. **127**, 162–169 (1954); these Rev. **15**, 604] the author proved that

$$A_p(u)\equiv(-1)^m W_p(u)\quad (\operatorname{mod} p),$$

where $W_p(u)=\sum_{s=0}^{m}\binom{m}{s}^2u^s$ $(p=2m+1)$, and p is a fixed prime. He now lets k^2 denote a root of $W_p(u)=0$ and defines the numbers $\alpha_m=A_m(k^2)$, $\beta_m=\beta_m(k^2)$, $\tau_m=\beta_m(k^2)/m$. Using for the most part real multiplication formulas of elliptic functions, he obtains the following results: (1) $\alpha_m\equiv 0 \pmod{p^r}$ for $m\geqq pr$; (2) $\beta_m\equiv 0 \pmod{p^{r-1}}$ for $m>pr$, $p-1\mid m$; (3) $\tau_m\equiv 0 \pmod{p^r}$ for $m>pr$, $p-1\nmid m$. The author also derives generalizations in various directions of these results. In conclusion he discusses the connection between $W_p(n)$ and the Legendre polynomial. *A. L. Whiteman.*

Carlitz, L. Dedekind sums and Lambert series. Proc. Amer. Math. Soc. **5**, 580–584 (1954).

In a previous paper [Pacific J. Math. **3**, 513–522 (1953); these Rev. **15**, 12] the author used the transformation formula for the Lambert series $G_p(x)=\sum_{n=1}^{\infty}n^{-p}x^n/(1-x^n)$, $x=e^{2\pi i\tau}$, p odd, to derive the functional equation

$$(1)\qquad f(h,k;\tau)=\tau^{p-1}f\left(-k,h;-\frac{1}{\tau}\right)+\frac{1}{\tau}(B+\tau B)^{p+1},$$

where f is a function which appears in the transformation formula for $G_p(x)$. [See these Rev. **15**, 12, for definition of f.] In this paper an elementary proof of (1) is given, based on a representation of f in terms of Eulerian numbers which was obtained in the earlier paper. *T. M. Apostol.*

Rademacher, Hans. Generalization of the reciprocity formula for Dedekind sums. Duke Math. J. **21**, 391–397 (1954).

Let a, b, c be positive integers, pairwise without common divisor and let $aa'\equiv 1 \pmod{bc}$, $bb'\equiv 1 \pmod{ca}$, $cc'\equiv 1 \pmod{ab}$. The author proves that

$$s(bc',a)+s(ca',b)+s(ab',c)=-\frac{1}{4}+\frac{1}{12}\left(\frac{a}{bc}+\frac{b}{ca}+\frac{c}{ab}\right),$$

where $s(a,b)$ is the Dedekind sum

$$\sum_{\mu=1}^{a-1}\left(\frac{\mu}{a}-\left[\frac{\mu}{a}\right]-\frac{1}{2}\right)\left(\frac{b\mu}{a}-\left[\frac{b\mu}{a}\right]-\frac{1}{2}\right).$$

The special case $c=c'=1$ is the reciprocity formula for Dedekind sums. *H. D. Kloosterman* (Leiden).

Carlitz, L. A note on generalized Dedekind sums. Duke Math. J. **21**, 399–403 (1954).

The author generalizes a result of Rademacher [see the preceding review]. Writing

$$f\left(\frac{r}{k}\right)=\frac{r}{k}-\left[\frac{r}{k}\right]-\frac{1}{2}+\frac{1}{2k}$$

and

$$s_n(h_1,\cdots,h_n;k)=\sum_{r_\nu \bmod k}f\left(\frac{r_1}{k}\right)\cdots f\left(\frac{r_n}{k}\right)f\left(\frac{r_1h_1+\cdots+r_nh_n}{k}\right)$$

(the h_i, k and n are integers; $n\geqq 1$), he proves that

$$\sum k_n^{n-2}s_{n-1}(k_1,\cdots,k_{n-1};k_n)=Z_n^{(n)}/n!\quad (n\geqq 2)$$

and

$$\sum k_n^{n-3}s_{n-2}(k_1k'_{n-1},\cdots,k_{n-2}k'_{n-1};k_n)=\sum_{r=1}^{n-1}(-1)^{n-r-1}Z_r^{(n)}/r!+(-1)^{n-1}\left(-1+\frac{1}{k_1\cdots k_n}\right),$$

where $k_1,\cdots,k_n$ are positive integers that are relatively prime in pairs, $k_ik_i'\equiv -1 \pmod{k_1\cdots k_{i-1}k_{i+1}\cdots k_n}$ and the summations on the left hand sides are extended over the cyclic permutations of $k_1,\cdots,k_n$. The $Z_r^{(n)}$ are certain numbers that are related to generalized Bernoulli numbers. The second formula reduces to Rademacher's result if $n=3$. *H. D. Kloosterman* (Leiden).

Selmer, Ernst S. A conjecture concerning rational points on cubic curves. Math. Scand. **2**, 49–54 (1954).

The author reports that using an electronic computor he has found solutions for all the equations for which he could give no decision in an earlier memoir [Acta Math. **85**, 203–362 (1951); these Rev. **13**, 13] except in two cases where he can prove insolubility by different criteria. This and other numerical evidence to be presented elsewhere later leads to interesting conjectures. If the L.H.S. of the Diophantine equation $x^3+y^3=az^3$ is factorised in the Eisenstein field of cube roots of 1 and coefficients are equated, there result a finite number of equations of the type $w^3=f(u,v)$ where $f(u,v)$ is a ternary cubic with rational integer coefficients. The number of these which are soluble is closely related to the number of generators of the group of rational solutions in the usual sense. By congruence considerations equations $w^3=f(u,v)$ may often be shown to be insoluble. One can now factorise $f(u,v)$ in the relevant cubic fields and so obtain new equations of the type $\phi(u',v',w')=0$, where ϕ is a ternary cubic with integer coefficients. These

can be again sometimes eliminated by congruence considerations ("second descent"). The author remarks that in a large number of cases an even number of generators is eliminated by the second descent and conjectures that this always happens. Indeed he conjectures that when a second descent exists the difference between the number of generators allowed by the first descent and that actually existing is even. He also shows that a second descent can be made with $Y^2=X^3-AX-B$ and makes similar conjectures.

J. W. S. Cassels (Cambridge, England).

Nagell, Trygve. Sur la division des périodes de la fonction $\wp(u)$ et les points exceptionnels des cubiques. Nova Acta Soc. Sci. Upsaliensis (4) 15, no. 8, 28 pp. (1953).

In previous papers the author has discussed the group Γ of exceptional points on a plane cubic curve $y^2=x^3-Ax-B$ of genus one over a field Ω by means of the well-known parametrization of the cubic in Weierstrassian elliptic functions [cf., e.g., same Acta (4) 14, no. 1 (1946), no. 3 (1947); these Rev. 9, 100]. He has considered subgroups G_n of Γ of various finite orders n; the difficult case of a G_{16} was examined earlier [ibid. 15, no. 6 (1952); these Rev. 14, 1010]. In the present paper he continues this investigation and obtains among other results: (1) Necessary and sufficient that a cubic over Ω should admit a G_{16} is that the cubic $y^2=4x^3-7x^2+4x$ should have on it a rational point (in Ω) distinct from 20 specified points; (2) when $\Omega=K(1)$, $K(i)$, a cubic of genus one cannot have a G_{16}; (3) there are infinitely many quadratic fields Ω (characterized by a necessary and sufficient condition) in which there exist cubics of genus one with a G_{16}. *J. Lehner* (Los Alamos, N. M.).

Bergman, Gösta. On the exceptional group of a Weierstrass curve in an algebraic field. Acta Math. 91, 113–142 (1954).

Let Ω be an algebraic field, and let A, B be two of its numbers satisfying $4A^3-27B^2\neq 0$. The plane elliptic curve (1) $y^2=x^3-Ax-B$ can then be represented parametrically by

$$x=\wp(u;4A,4B),\quad y=\tfrac{1}{2}\wp'(u;4A,4B);$$

we denote by ω, ω' a primitive pair of periods of the $\wp$-function. A point u of the curve (1) is said to be a point in Ω if its coordinates (x,y) belong to Ω; it is called an exceptional point of order q if u is commensurable with a period and if q is the smallest natural number such that $qu\equiv 0 \pmod{\omega,\omega'}$. It is well known that the u-values of the exceptional points in Ω form an abelian group, called the exceptional group in Ω of the curve (1).

This paper, by a deep arithmetical analysis of the elliptic functions which express the coordinates x, y or an exceptional point, and of the relations among the points νu $(\nu=0,\pm1,\pm2,\cdots)$, proves that the order q of an exceptional point in Ω of a given plane elliptic curve is bounded, and actually determines an upper bound for q.

For this purpose, it is first of all remarked that, if α and $\mathfrak{p}$ denote a number $\neq 0$ and a prime ideal in Ω, the curve (1) and $y^2=x^3-A\alpha^4x-B\alpha^6$ are equivalent, so that A and B can be supposed to be integers mod $\mathfrak{p}$. Hence it is shown that, if $\mathfrak{p}$ is a divisor of 2, then the (x,y)-coordinates of any exceptional point in Ω, not at infinity, are integers mod $\mathfrak{p}$. Denote by N the norm of $\mathfrak{p}$, by $q=2^\lambda t$ the order of an exceptional point in Ω (where $\lambda\geq 0$, and t is an odd number ≥ 1), and suppose $\mathfrak{p}^m\|2$; if $\mathfrak{p}^4|A$, then $\mathfrak{p}^6\nmid B$; if $\mathfrak{p}^m\|B$, then $\mathfrak{p}|A$ (here $\mathfrak{p}^n\|\alpha$ expresses that n is the integer such that $\alpha=\mathfrak{p}^n\mathfrak{a}/\mathfrak{b}$, where $\mathfrak{a}$ and $\mathfrak{b}$ are convenient integral ideals in Ω and $\mathfrak{p}\nmid\mathfrak{ab}$). A limit for q which depends only on Ω is then given by proving the following inequalities:

$$t\leq 2N+1;\quad 2^\lambda\leq \max[2N,4(3(m+1))^{1/2}];$$
$$q\leq \max[2N+1,3(6(m+1))^{1/2}],\quad \text{if}\quad t=3;$$
$$q\leq (2N+1)(2(m+1))^{1/2},\quad \text{if}\quad t\geq 5.$$

Similar inequalities are obtained in the remaining case, i.e. when $\mathfrak{p}\nmid A$; $\mathfrak{p}^m\|B$.

These results imply a new proof of a theorem by A. Weil [Acta Math. 52, 281–315 (1929)], according to which the exceptional group in an algebraic field is always finite, and also make it possible to find the points of this group by a regular process. *B. Segre* (Rome).

Mautner, F. I. On congruence characters. Monatsh. Math. 57, 307–316 (1954).

The author remarks the following topologico-analytic method in number theory. Let U_p be the multiplicative unit group of the p-adic number field with compact topology and take the direct product U of U_p's for all finite primes p. Then we can embed the multiplicative group ρ^* of the rational number field isomorphically in U. This mapping u is determined by $q\to u(q)=\{\cdots,u(q)_p,\cdots\}\in U$ for every prime q, where $u(q)_p=q$ for $p\neq q$; $=\epsilon_q$ for $p=q$ (by an arbitrary element ϵ_q of U_q). The main result is that the sequences $\{u(n)\}$ of all integers and $\{u(q)\}$ of all prime numbers with respect to their natural orders are uniformly distributed in the compact group U. The following is an application. Let c_n be a sequence of complex numbers for which $\Phi(u(n))=c_n$ defines a uniformly continuous function on $u(\rho^*)$. Then the Dirichlet series $\Psi(s)=\sum_{n=1}^\infty c_n n^{-s}$ converges for $\mathrm{Re}(s)>1$ and $\lim_{s\to 1+0}(s-1)\Psi(s)=\int_U\Psi(u)\,d\mu(u)$ holds where μ is the Haar measure on U with $\mu(U)=1$. For an arbitrary algebraic number field K the same method can be applied by taking the factor group of the idèle class group C_K by the component D_K of the unity as U, and by taking the group of ideals of K instead of ρ^*. *Y. Kawada.*

Košlyakov, N. S. Investigation of some questions of the analytic theory of a rational and quadratic field. I. Izvestiya Akad. Nauk SSSR. Ser. Mat. 18, 113–144 (1954). (Russian)

There are a number of identities in analysis such as the partial fraction expansion of $1/(e^{2\pi x}-1)$ and the functional equations for the one-dimensional theta-functions which are analytically equivalent via the Mellin transform to the functional equation for the Riemann zeta-function. In this paper the author considers the zeta-function associated with a quadratic field, $\zeta_Q(s)=\sum_a 1/Na^s=\sum_{n=1}^\infty F(n)/n^s$, and systematically derives the analogues of the above identities for the various classes of fields that occur. The basic functions are the Voronoi functions of order two, which can be expressed in terms of Bessel functions, and the fundamental technique is a combination of the Mellin inversion formula and the functional equation for $\zeta_Q(s)$. *R. Bellman.*

Košlyakov, N. S. Investigation of certain questions of the analytic theory of a rational and quadratic field. II. Izvestiya Akad. Nauk SSSR. Ser. Mat. 18, 213–260 (1954). (Russian)

In the first paper of this series the author considered functional equations for expressions of the form $\sum_{n=1}^\infty F(n)f(n)$

where $f(n)$ had various special forms. In this paper he derives the analogue of the Abel-Cauchy-Plana summation formula for the above sum. Furthermore, he obtains analogues of the Bernoulli numbers and polynomials for quadratic fields. The paper contains a number of interesting and elegant identities. *R. Bellman.*

Herrmann, Oskar. Eine metrische Charakterisierung eines Fundamentalbereichs der Hilbertschen Modulgruppen. Math. Z. **60**, 148–155 (1954).

Let K be a totally real field of degree n, and let $\mathfrak{G}$ be a group of totally positive units. Then the Hilbert group Γ is defined as the group of all transformations

$$\tau_k' = (\alpha^{(k)}\tau_k + \beta^{(k)})/(\gamma^{(k)}\tau_k + \delta^{(k)})$$

in the domain $\operatorname{Im}\tau_1 > 0, \cdots, \operatorname{Im}\tau_n > 0$. Here $\alpha, \beta, \gamma, \delta$ are integers of K, $\alpha\delta - \beta\gamma \varepsilon \mathfrak{G}$, and (k) denotes kth conjugate. The author determines a fundamental domain for Γ which is essentially simpler than the one of Maass [S.-B. Heidelberger Akad. Wiss. **1940**, no. 2; these Rev. **2**, 213]. It is determined by the set of inequalities:

(i) $\mathbf{N}|\gamma\tau+\delta|^2 \geqq 1$ for all pairs $\gamma \varepsilon K$, $\delta \varepsilon K$, $(\gamma, \delta) = 1$;
(ii) $\mathbf{S}\log \epsilon(\log \epsilon + 2\log \operatorname{Im}\tau) \geqq 0$ for all $\epsilon \varepsilon \mathfrak{G}$;
(iii) $\mathbf{S}\mu(\mu + 2\operatorname{Re}\tau) \geqq 0$ for all integers $\mu \varepsilon K$.

($\mathbf{N}$ and $\mathbf{S}$ denote norm and trace, and in these formulas $\tau_1, \cdots, \tau_n$ are formally interpreted as the conjugates of τ.) It is also shown that the domain can already be described by a finite number of these inequalities. *N. G. de Bruijn* (Amsterdam).

Siegel, Carl Ludwig. A simple proof of

$$\eta(-1/\tau) = \eta(\tau)\sqrt{\tau/i}.$$

Mathematika 1, 4 (1954).

The author gives a short and extremely elegant proof of the equation in the title, which is a special case of the difficult transformation formula first proved in generality by Dedekind and Hermite. Taking logarithms and expanding in power series, one finds that the formula in question is equivalent to

$$\pi i\frac{\tau+\tau^{-1}}{12} + \frac{1}{2}\log\frac{\tau}{i} = \sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{1}{e^{-2\pi ik\tau}-1} - \frac{1}{e^{2\pi ik/\tau}-1}\right).$$

Let $f(z) = \tau^{-1}\cot z \cot z$, $\nu = (n+\frac{1}{2})\pi$, $n = 0, 1, 2, \cdots$, and integrate $z^{-1}f(\nu z)$ over the contour C of the rhombus with vertices at $1, \tau, -1, -\tau$ in the z-plane. Then by the residue theorem

$$\pi i\frac{\tau+\tau^{-1}}{12} + \int_C f(\nu z)\frac{dz}{8z} = \sum_{k=1}^{n}\frac{1}{k}\left(\frac{1}{e^{-2\pi ik\tau}-1} - \frac{1}{e^{2\pi ik/\tau}-1}\right).$$

But as $n \to \infty$, $f(\nu z)$ has on the sides of the rhombus the limiting values $1, -1, 1, -1$. This gives

$$\lim_{n\to\infty}\int_C f(z)\frac{dz}{z} = 4\log\tau/i$$

and completes the proof. *J. Lehner.*

Tatuzawa, Tikao. The approximate functional equation for Dirichlet's L-series. Jap. J. Math. **22** (1952), 19–25 (1953).

The main result is Theorem 2: If $0<\sigma<1$, $x \geqq \sqrt{k}$, $y \geqq \sqrt{k}$, $2\pi xy = kt$ $(s = \sigma + it)$, and $\chi(n)$ is a Dirichlet character mod k, then

$$L(s,\chi) = \sum_{n\leqq x}\chi(n)n^{-s} + 2(2\pi)^{s-1}k^{-s}\Gamma(1-s)\sum_{n\leqq y}\left(C(n)\sin\frac{\pi s}{2} + S(n)\cos\frac{\pi s}{2}\right)n^{s-1} + O(kx^{-\sigma}) + O(ky^{\sigma-1}(kt)^{\frac{1}{2}-\sigma}),$$

where

$$C(n) = \sum_{l=1}^{k}\chi(l)\cos\frac{2\pi ln}{k}, \quad S(n) = \sum_{l=1}^{k}\chi(l)\frac{\sin 2\pi ln}{k},$$

and O implies an absolute constant and an inequality valid for all stated values of the variables. The emphasis is on uniformity in k. Tchudakoff [Ann. of Math. (2) **48**, 515–545 (1947); these Rev. **9**, 11] had already obtained a result of this nature for primitive χ, when the second sum takes a simpler form in terms of a partial sum of the Dirichlet series for $L(1-s, \bar{\chi})$. The argument here follows the lines of the Hardy-Littlewood 'real variable' proof of the approximate functional equation for $\zeta(s)$ [Proc. London Math. Soc. (2) **29**, 81–97 (1929)]. It is stated that, when χ is primitive, the first factor k in the error terms may be reduced 'by a slight careful estimation' to $k^{\frac{1}{2}}\log(k+1)$. There are a few minor errors and some troublesome obscurities. Thus Lemma 4 is quoted from Hardy and Littlewood in an extended form, but without comment on the extension (particularly the extension from $\frac{1}{2} \leqq \sigma \leqq \frac{3}{2}$ to $0<\sigma<2$). Theorem 1 involves an infinite series, but the enunciation contains no reference to convergence or range of validity. The theorem is first proved for $-1<\sigma<0$ (the reader being left to supply his own justification of term-by-term limit operation), and could be extended at once by analytic continuation to $\sigma<0$ (the half-plane of absolute convergence). If something more than this is meant, the only clue is the cryptic sentence, 'By a slight modification we can deduce this functional equation for an arbitrary half plane'. However, Theorem 1 is not used in the sequel; it serves merely to prepare the reader for the form of the second sum in Theorem 2. *A. E. Ingham* (Cambridge, England).

Knobloch, Hans-Wilhelm. Über Primzahlreihen nebst Anwendung auf ein elementares Dichteproblem. Abh. Math. Sem. Univ. Hamburg **19**, no. 1–2, 1–13 (1954).

For $s>1$ let $F(s; m, a)$ denote the Dirichlet series $\sum r^{-1}p^{-rs}$, summed over all prime powers p^r, $r \geqq 1$, such that $p^r \equiv a \pmod m$; let $f(s; m, a)$ denote the similar series with the additional condition $p^r > m$. The main result of the paper is that if $s_0 > 1$ is fixed, then for every $\epsilon > 0$, there is a constant $C_\epsilon > 0$, independent of s, m, a, such that $f(s; m, a) - f(s_0; m, a) < C_\epsilon m^{\epsilon-1}\log\{s/(s-1)\}$ holds uniformly for $1 < s \leqq s_0$. The proof uses Dirichlet L-series and only elementary real variable theory.

As an application it follows that, in the infinite series $\sum\mu(m)F(s; m^2, 1)/\log\{s/(s-1)\}$, the limit as $s\to 1+0$ may be taken term by term. This, in turn leads to the result that the set of primes p for which $p-1$ is quadratfrei has Dirichlet density [see, e.g., Hasse, Vorlesungen über Zahlentheorie, Springer, Berlin, 1950, pp. 223–226; these Rev. **14**, 534] given by $\prod_p\{1-1/p(p-1)\}$. A note added in proof by the referee points out that this last result is contained in a paper by Mirsky [Amer. Math. Monthly **56**, 17–19 (1949); these Rev. **10**, 431], but that the functions $f(s; m, a)$ are not considered there. *R. D. James* (East Lansing, Mich.).

Lambek, J., and Moser, L. Inverse and complementary sequences of natural numbers. Amer. Math. Monthly 61, 454–458 (1954).

The authors generalize Beatty's theorem and arrive at a reformulation of the general inversive algorithm of Viggo Brun. Two sequences

$$f:\ f(1), f(2), \cdots,$$
$$g:\ g(1), g(s), \cdots,$$

are said to be mutually inverse in case exactly one of the inequalities $f(m)<n$, $g(n)<m$ holds whenever a pair of positive integers (m, n) is chosen. For a sequence f to have an inverse it is necessary and sufficient that f be non-decreasing. Associated with f and g are the functions or sequences F and G defined by $F(m)=m+f(m)$, $G(n)=n+g(n)$. For f and g to be inverse it is necessary and sufficient that F and G be complementary, that is, exactly one of the equations $F(x)=N$, $G(x)=N$ has a solution x for every choice of positive integer N. This is a generalization of Beatty's theorem to the effect that if x and y are irrationals for which $x^{-1}+y^{-1}=1$ then $[nx]$ and $[ny]$ are complementary.

The Brun algorithm is derived in terms of a sequence and its complementary sequence. The authors' statement may be put as follows. Let $p_1<p_2<p_3<\cdots$ be a sequence of positive integers. Let the number of p's not exceeding x be denoted by $\pi(x)$. Let $q_1<q_2<q_3<\cdots$ be the set complementary to the p's. Define recursively $F_k(n)$ by

$$F_0(n)=n, \quad F_k(n)=n+\pi(F_{k-1}(n)),$$

then $\lim_{k\to\infty} F_k(n)=q_n$. Thus if p_i is the ith prime, the nth non-prime is $n+\pi(n+\pi(n+\pi(n+\cdots)))$. There are seven examples in the paper. The reader is warned that in the last two pages the authors have tacitly assumed that the sequences involved are strictly increasing.

D. H. Lehmer (Berkeley, Calif.).

Duparc, H. J. A. A short proof of a property of Ward on recurring series. Math. Centrum Amsterdam. Rapport **ZW 1953-017**, 2 pp. (1953).

M. Ward's theorem [Proc. Nat. Acad. Sci. U. S. A. **19**, 914–916 (1933)] is concerned with sequences $\{u_n\}$ of rational numbers satisfying a linear difference equation of order N with rational coefficients. If a particular solution is such that

$$u_a=u_{a+b}=u_{a+2b}=\cdots=u_{a+Nb}\neq 0,$$

where a and b are fixed integers and if the characteristic polynomial $f(x)$ is irreducible in the rational field, then $f(x)$ is a cyclotomic polynomial and every solution of the difference equation is periodic. The present author generalizes the theorem to the case where the rational field is replaced by a ring satisfying certain conditions. *W. Ledermann.*

Ananda-Rau, K. On the representation of a number as the sum of an even number of squares. J. Madras Univ. Sect. B. **24**, 61–89 (1954).

Certain arithmetical formulae for expressing the number of representations of n as a sum of an even number of squares were enunciated by Ramanujan (without proof) in the form of analytical identities involving elliptic theta-functions and series of the Lambert type [Collected papers of Srinivasa Ramanujan, Cambridge, 1927, paper no. 18, pp. 136–162]. Mordell later proved Ramanujan's formulae [Quart. J. Pure Appl. Math. **48**, 93–104 (1918)], by applying the theory of modular invariants. In this paper the author employs the theory of elliptic functions to give new proofs of these identities, and asserts that his methods lead to other interesting formulas which he hopes to consider in a future paper. *T. M. Apostol* (Pasadena, Calif.).

Ramasarma, B. V. Partitions of zero into 4 cubes. Math. Student **22**, 102–103 (1954).

Meinardus, Günter. Asymptotische Aussagen über Partitionen. Math. Z. **59**, 388–398 (1954).

The author applies a saddle-point method to obtain the asymptotic value of the power-series coefficients $r(n)$ of the function $f(x)=\prod_{n=1}^{\infty}(1-x^n)^{-a_n}$, where a_n are real, non-negative numbers. His formula is

$$(1)\quad r(n)=Cn^{\kappa}\exp\left\{n^{\alpha/(\alpha+1)}\left(1+\frac{1}{\alpha}\right)\times(A\Gamma(\alpha+1)\zeta(\alpha+1))^{1/(\alpha+1)}\right\}\cdot(1+O(n^{-k_1})),$$

where α is the convergence abscissa of $D(s)=\sum a_n n^{-s}$, A is its residue at $s=1$, and κ, C, k_1 are explicitly given in terms of α, $D(0)$, $D'(0)$. The validity of (1) requires that $D(s)$ and $\sum_1^{\infty} a_n x^n$ shall satisfy certain function-theoretic conditions.

The author closes the paper with certain examples, including one in which $r(n)$ is the number of partitions of n into summands $n_1\equiv a \pmod{n}$, $(a, n)=1$, and so obtains the first terms of the Hardy-Ramanujan-Rademacher series. The reviewer remarks that he could equally well have considered the case in which $n_i\equiv\pm a \pmod{n}$, and so derived the first terms of the convergent series obtained for special values of a and n by Niven [these Rev. **1**, 201], Haberzetle [these Rev. **3**, 69], the reviewer [these Rev. **3**, 166], and Livingood [these Rev. **6**, 259].

The author points out that his method does not require the assumption of monotonicity for $r(n)$, which is needed when a Tauberian approach is used [e.g., Ingham, Ann. of Math. (2) **42**, 1075–1090 (1941); these Rev. **3**, 166].

J. Lehner (Los Alamos, N. M.).

Haselgrove, C. B., and Temperley, H. N. V. Asymptotic formulae in the theory of partitions. Proc. Cambridge Philos. Soc. **50**, 225–241 (1954).

The authors apply a method of contour integration to obtain asymptotic results about $p_m(n)$, the number of partitions of an integer n into m parts selected from the sequence of integers $0<\lambda_1<\lambda_2<\cdots\to\infty$; the generating function $G(x, z)=\prod_{r=1}^{\infty}(1-xz^{\lambda_r})^{-1}$ is used. Their principal result is that if $\sum\lambda_r^{-2}$ converges and if $\psi(\omega)=\sum e^{-\lambda_r\omega}$, $\operatorname{Re}\omega>0$, satisfies certain function-theoretic restrictions, then

$$p_m(n)/p(n)\sim\xi F((m-m_0)\xi), \quad n\to\infty.$$

In this formula, $p(n)$ is the number of unrestricted partitions of n, ξ is the root of a transcendental equation involving n and $\psi(\omega)$, $m_0=\sum_r(e^{\lambda_r\xi}-1)^{-1}$, and $F(y)$ has the properties of a probability density function with mean at $y=0$, which is given explicitly as the inverse Laplace transform of the function $\prod_r(1+\alpha/\lambda_r)^{-1}e^{\alpha/\lambda_r}$. By means of this formula and some others involving differences of the $p_m(n)$, the authors are able to verify known results on partitions into m parts, into powers, and into primes, and to prove the conjecture of Auluck, Chowla, and Gupta [J. Indian Math. Soc. (N.S.) **6**, 105–112 (1942), p. 105; these Rev. **4**, 211] that $p_m(n)$ attains its maximum value for at most two consecutive values of m when n is large and fixed.

The method employed appears to be the most powerful of the methods which do not use the Farey dissection of the circle. *J. Lehner* (Los Alamos, N. M.).

Davenport, H., and Watson, G. L. The minimal points of a positive definite quadratic form. Mathematika 1, 14–17 (1954).

Let $q(x_1, \cdots, x_n)=q(\mathbf{x})$ be a positive definite n-ary quadratic form with real coefficients, and let $\mathbf{x}^{(1)}, \cdots, \mathbf{x}^{(n)}$ be points at which $q(\mathbf{x})$ assumes its successive minima, so that for each k, $q(\mathbf{x}^{(k)})$ is the smallest value assumed by q for any $\mathbf{x}$ linearly independent of $\mathbf{x}^{(1)}, \cdots, \mathbf{x}^{(k-1)}$. Let N be the determinant of the coordinates of $\mathbf{x}^{(1)}, \cdots, \mathbf{x}^{(n)}$, and let $N(q)=\min |N|$ for all such sets $(\mathbf{x}^{(1)}, \cdots, \mathbf{x}^{(n)})$. Improving a result of the second author (apparently not published), it is shown that if n is large, there is a form q such that $N(q)>\pi^{-n/2}\Gamma(1+\frac{1}{2}n)$. *W. J. LeVeque.*

LeVeque, W. J. On asymmetric approximations. Michigan Math. J. 2, 1–6 (1954).

It is known that every irrational ξ has infinitely many approximations u/v such that

$$(*)\qquad -\frac{\tau}{\alpha v^2}<\xi-\frac{u}{v}<\frac{1}{\alpha v^2},$$

where u, v are rational integers and

$$\alpha=\max \{(1+4\tau)^{1/2}, (\tau^2+4\tau)^{1/2}\}$$

[B. Segre, Duke Math. J. 12, 337–365 (1945); these Rev. 6, 258]. When $\tau=1$ a classical theorem states that indeed one of three consecutive convergents p_n/q_n to ξ satisfies (*) and N. Negoescu alleged that this contines to hold in the general case [Acad. Repub. Pop. Române. Bul. Şti. A. 1, 115–117 (1949); these Rev. 13, 630]. The author disproves this with an example but shows that one of p_{n-1}/q_{n-1}, p_n/q_n, p_{n+1}/q_{n+1} satisfies (*) with $(\tau^2+4\tau)^{1/2}$ if n is odd but with $(1+4\tau)^{1/2}$ if n is even. *J. W. S. Cassels* (Cambridge, England).

Gonçalves, J. Vicente. Sur les fractions continues réelles. Univ. Lisboa. Revista Fac. Ci. A. Ci. Mat. (2) 2, 297–335 (1952).

The first part of the paper contains certain classical results of Galois and Lagrange for periodic simple continued fractions. In part two the diophantine equation $x^2-By^2=\pm C$ is considered, where B, C are rational and $B^{1/2}$ is irrational. It is noted that a fraction h/l is an approximant of the expansion for $B^{1/2}$ if and only if $|h^2-Bl^2|\leq B^{1/2}+|Bll'-hh'|$, where h'/l' is the next to last approximant of the expansion for h/l. This leads to a classification of solutions (h, l) of the diophantine equation into a first and a second category according as the inequality $C\leq B^{1/2}+|Bll'-hh'|$ is or is not satisfied. Restatement of this classification leads to necessary and sufficient conditions in order that (h, l) be a solution of a prescribed category. In part three a convergence theorem for continued fractions with real elements is given; the theorem includes that of Tietze. *W. T. Scott.*

ANALYSIS

Agmon, Shmuel. The relaxation method for linear inequalities. Canadian J. Math. 6, 382–392 (1954).

If $\sum_{j=1}^{n} a_{ij}x_j+b_i\geq 0$ $(i=1, 2, \cdots, m)$ is a consistent system of linear inequalities, the non-empty solution set A is the product of the m closed half-spaces H_i $(i=1, 2, \cdots, m)$ defined in the n-space E_n of the indeterminates by the individual inequalities of the system. Denote by π_i the hyperplane bounding H_i $(i=1, 2, \cdots, m)$ and let p be an arbitrary point of E_n. If p non-ϵ A, consider the point $p_1=p+\lambda(q-p)$, where q is the orthogonal projection of p on a farthest π_i, and λ is a positive constant at most 2. If p_1 non-ϵ A, a similar process yields p_2, and a continuation of the procedure (with λ fixed) either terminates in a point of A or an infinite sequence of points $p_1, \cdots, p_n, \cdots$ is obtained, all outside A. The process is known as the relaxation method for linear inequalities. The principal result of this paper is the fundamental one of justifying the method by proving that if $0<\lambda<2$, an infinite sequence yielded by the process converges to a point of A. *L. M. Blumenthal* (Leiden).

Motzkin, T. S., and Schoenberg, I. J. The relaxation method for linear inequalities. Canadian J. Math. 6, 393–404 (1954).

This paper presents an elegant proof of the fundamental theorem established in the article reviewed above and obtains new results for the case $\lambda=2$ (the reflection method). Denoting by A the solution set of a consistent system of m linear inequalities in n indeterminates, it is shown that the reflection method always terminates when $\dim A=n$, while if $\dim A=r<n$ then either the sequence $\{p_\nu\}$ terminates or else there is a number ν_0 such that for $\nu\geq\nu_0$ the points p_ν are on a spherical surface S_{n-r-1} having the r-dimensional subspace containing A as axis (i.e., L_r is the locus of points equidistant from the points of S_{n-r-1}). The concluding section of the paper is concerned with establishing analogous theorems for the reflection process when the (infinite) set of half-spaces considered all contain a given bounded, closed, and convex subset A (which it is desired to enter) and their boundaries are supporting hyperplanes of A.

L. M. Blumenthal (Leiden).

Green, J. W. Recent applications of convex functions. Amer. Math. Monthly 61, 449–454 (1954).

Expository paper.

Dinghas, Alexander. Zur Abschätzung arithmetischer Mittel reeller Zahlen durch Differenzenprodukte derselben. Rend. Circ. Mat. Palermo (2) 2 (1953), 177–202 (1954).

The inequality of Schur [Math. Z. 1, 377–402 (1918)] and Siegel [Ann. of Math. (2) 46, 302–312 (1945); those Rev. 6, 257], between the elementary symmetric functions $S_1(x)$ and $S_n(x)$ of positive numbers $x_1, \cdots, x_n$, improves the inequality between the geometric and arithmetic means. The author now extends the inequality to other elementary symmetric functions S_r $(r=2, \cdots, n)$. Applications are made, for instance, to the sharpening of the Hölder and Chebyshev inequalities. *E. F. Beckenbach.*

Jecklin, Heinrich. Sull'interpolazione iperbolica. Giorn. Ist. Ital. Attuari 15 (1952), 250–260 (1953).

Malliavin, Paul. Théorèmes d'adhérence pour certaines séries de Dirichlet. Procédés d'extrapolation en analyse fonctionnelle. C. R. Acad. Sci. Paris 239, 20–22 (1954).

The author states further results [same C. R. 238, 2481–2483 (1954); these Rev. 15, 942], in particular on sets of

uniqueness for meromorphic functions, the order of entire functions represented by Dirichlet series, the singular points of Dirichlet series, an abstract form of Müntz's theorem, and generalizations to n dimensions, and to certain operators, of properties of quasi-analyticity and inequalities among maxima of derivatives which were known in the one-dimensional case. *R. P. Boas, Jr.* (Evanston, Ill.).

Četković, Vida. Sur la continuité d'un ensemble de zéros réels des dérivées d'une classe de fonctions. Bull. Soc. Math. Phys. Serbie **5**, no. 3–4, 111–113 (1953). (Serbo-Croatian. French summary)

Calculus

***Gellerstedt, Sven. 800 övningsuppgifter i matematik för universitet och högskolor. [800 problems in mathematics for universities.]** Almqvist & Wiksell, Stockholm, 1954. v+200 pp. 12.50 kr.

The problems in this collection are largely taken from examinations set at the University of Uppsala for students of university-level mathematics, although a few problems have been drawn from other sources. The fields represented in these problems are: algebra (largely divisibility and solutions of polynomial equations); the differential calculus (inequalities, limits, geometric applications); the integral calculus (computation of indefinite and definite integrals, curve lengths, areas, volumes, and differential equations); sequences, series, and products; plane and solid analytic geometry. A list of answers and hints is provided. Solution of many of the problems requires considerable ingenuity as well as computational technique. Much must be said for an educational system producing students with such highly developed skills. On the other hand, modern developments are not mentioned anywhere: the reviewer finds nothing in this book that would have puzzled Euler. The book is, as its title indicates, written at a much more elementary level than Pólya and Szegö's well-known collection, Aufgaben und Lehrsätze aus der Analysis [Bd I, II, 2. Aufl., Springer, Berlin, 1954; these Rev. **15**, 512]. *E. Hewitt.*

Corominas, Ernest. Contribution à la théorie de la dérivation d'ordre supérieur. Bull. Soc. Math. France **81**, 177–222 (1953).

The author systematically investigates the extension, to higher derivatives and difference quotients, of the classical theorems of the differential calculus, including the theorems of Rolle, Cauchy, and Lagrange; some applications are indicated. *E. F. Beckenbach* (Los Angeles, Calif.).

Rosada, Giorgio. Generalizzazione della formula di Taylor per le funzioni di una variabile. Period. Mat. (4) **32**, 77–80 (1954).

Farr, Harold K. Discussion of "The meaning of the vector Laplacian." J. Franklin Inst. **258**, 213–214; discussion 215–216 (1954).

Discussion of a recent paper by Moon and Spencer [same J. **256**, 551–558 (1953); these Rev. **15**, 311].

Theory of Sets, Theory of Functions of Real Variables

Fodor, G., und Ketskeméty, I. Über eine Eigenschaft der singulären Kardinalzahlen. Colloquium Math. **3**, 39–40 (1954).

A direct proof of the following well-known consequence of the König-Jourdain-Zermelo theorem: If $\aleph_\alpha$ is singular and ω_β is the smallest initial number cofinal with ω_α, then $\aleph_\alpha^{\aleph_\beta} > \aleph_\alpha$. *F. Bagemihl* (Princeton, N. J.).

Dushnik, Ben. Upper and lower bounds of order types. Michigan Math. J. **2**, 27–31 (1954).

Let α and β be order types, A and B be ordered sets with $\bar{A}=\alpha$, $\bar{B}=\beta$. If A is similar to a subset of B, write $\alpha\leqq\beta$; if, at the same time, B is not similar to any subset of A, write $\alpha<\beta$. If neither $\alpha\leqq\beta$ nor $\beta\leqq\alpha$, write $\alpha\|\beta$. Suppose that $\alpha\leqq\gamma$ and $\beta\leqq\gamma$; if, for every δ satisfying $\alpha\leqq\delta$ and $\beta\leqq\delta$, we have either $\gamma<\delta$ or $\gamma\|\delta$, call γ a least upper bound of α and β. Define greatest lower bound analogously. If α and β are transfinite ordinals, then α and β^* have no greatest lower bound. If $\alpha=\omega\cdot r+m$, and $\beta=\omega\cdot s+n$, where r and s are natural numbers and m and n are non-negative integers, then the only least upper bounds of α and β^* are the order types

$$n+\omega^*\cdot b_1+\omega\cdot a_1+\omega^*\cdot b_2+\omega\cdot a_2+\cdots+\omega^*\cdot b_t+\omega\cdot a_t+m,$$

where $t, a_1, \cdots, a_t, b_1, \cdots, b_t$ are natural numbers (except that b_1 or a_t may be 0) satisfying $a_1+\cdots+a_t=r$, $b_1+\cdots+b_t=s$. *F. Bagemihl* (Princeton, N. J.).

Neumer, Walter. Zur Konstruktion von Ordnungszahlen. III. Math. Z. **60**, 1–16 (1954).

After deriving some rules of operation for the "free" algorithm developed in the second paper [Math. Z. **59**, 434–454 (1954); these Rev. **15**, 689] of this series, the author applies them to show that each of two constructive algorithms based on simple operators comprehends the same initial segment of the second number class as the constructive algorithm developed in the first paper [Math. Z. **58**, 391–413 (1953); these Rev. **15**, 512] of this series. There is also a discussion of several methods of defining new operators in terms of given ones. *F. Bagemihl.*

Neumer, Walter. Über Mischsummen von Ordnungszahlen. Arch. Math. **5**, 244–248 (1954).

If $\alpha=\omega\alpha'+a$ $(a<\omega)$ and $\beta=\omega\beta'+b$ $(b<\omega)$ are ordinal numbers, let $\alpha\breve{+}\beta$ be their Hessenbergian natural sum, and $\alpha\hat{+}\beta$ be $\max(\alpha,\beta)$ if $\alpha'\neq\beta'$, $\omega\alpha'+a+b$ if $\alpha'=\beta'$. Let $\alpha_1, \cdots, \alpha_n$ be a finite number of nonzero ordinal numbers, and let $A_1, \cdots, A_n$ be mutually exclusive well-ordered sets such that $\bar{A}_\nu=\alpha_\nu$ $(\nu=1, \cdots, n)$. Order $\bigcup_{\nu=1}^n A_\nu$ in such a way that the original order of each A_ν $(\nu=1, \cdots, n)$ is preserved; the order type of the resulting set is an ordinal number. The largest and smallest ordinal numbers obtainable in this manner from $\alpha_1, \cdots, \alpha_n$ exist and are equal to $\alpha_1\breve{+}\cdots\breve{+}\alpha_n$ [for $n=2$: Carruth, Bull. Amer. Math. Soc. **48**, 262–271 (1942); these Rev. **3**, 225], $\alpha_1\hat{+}\cdots\hat{+}\alpha_n$, respectively. *F. Bagemihl* (Princeton, N. J.).

Rado, R. The minimal sum of a series of ordinal numbers. J. London Math. Soc. **29**, 218–232 (1954).

If φ and ψ $(\varphi<\psi)$ are ordinal numbers, define the interval $[\varphi, \psi)$ to be the set of ordinals τ such that $\varphi\leqq\tau<\psi$. For every ρ, every $\nu\leqq\rho$, and every nonzero $\kappa<\omega_{\rho+1}$ introduce

the following "fundamental intervals":

$$I=[0, \omega_0), \quad I(\rho, \kappa)=[\omega_\rho^\kappa, \omega_\rho^\kappa+\omega_0),$$
$$I(\rho, \kappa, \nu)=[\omega_\rho^\kappa+\omega_\nu, \omega_\rho^\kappa+\omega_{\nu+1}) \quad \text{if} \quad \nu<\rho,$$
$$I(\rho, \kappa, \nu)=[\omega_\rho^\kappa+\omega_\nu, \omega_\rho^{\kappa+1}) \quad \text{if} \quad \nu=\rho.$$

Let $s(\beta, \beta)=0$ for every ordinal β; for $\alpha<\beta$, define $s(\alpha, \beta)$ to be the least sum obtainable by arranging the ordinals of $[\alpha, \beta)$ in a well-ordered sequence and adding them in that order; and let $s(\beta)=s(0, \beta)$ for every β. Then every ordinal number lies in exactly one fundamental interval, and $s(\beta)$ is equal to $\frac{1}{2}\beta(\beta-1)$ for $\beta \in I$, $\beta(-\omega_\rho^\kappa+\beta)+\omega_\rho^\kappa$ for $\beta \in I(\rho, \kappa)$, and $\omega_\rho^\kappa\omega_\nu$ for $\beta \in I(\rho, \kappa, \nu)$. The function $s(\rho)$ is nondecreasing, and is continuous [in the sense of Sierpiński, Fund. Math. **38**, 204–208 (1951); these Rev. **13**, 828] everywhere except at $\omega_\rho^\kappa+\omega_{\mu+1}$ $(0<\kappa<\omega_{\rho+1}, \mu<\rho)$; for each such point ξ of discontinuity, there exist ordinals φ, ψ with $\varphi<\xi<\psi$, such that $s(\beta)$ is constant in $[\varphi, \xi)$ and in $[\xi, \psi)$, whereas $-s(\beta)+s(\xi)=s(\xi)$ for $\beta<\xi$. The function $s(\alpha, \beta)$ is equal to $\frac{1}{2}(\alpha+\beta-1)(\beta-\alpha)$ if $\alpha\leqq\beta<\omega_0$, $\alpha(-\alpha+\beta)$ if $-\alpha+\beta<\omega_0\leqq\alpha$, $\alpha\omega_\rho$ if $|-\alpha+\beta|=\aleph_\rho$ and $\beta\leqq\alpha\omega_\rho$, and $s(\beta)$ if α and β do not belong to the same fundamental interval; this includes all cases in which $\alpha\leqq\beta\leqq\alpha\omega_0$. [Misprint: in the statement of Theorem 1 on p. 220, insert "$+\omega_\rho^\kappa$" after "$x(x-\omega_\rho^\kappa)$".]

F. Bagemihl (Princeton, N. J.).

Eyraud, Henri. Les récurrences et le problème du transfini. Ann. Univ. Lyon. Sect. A. (3) **16**, 5–24 (1953).

Cf. Eyraud, Cahiers Rhodaniens **5**, 19–26 (1953); these Rev. **15**, 858. *F. Bagemihl* (Princeton, N. J.).

Eyraud, Henri. Les noyaux de divergence. Ann. Univ. Lyon. Sect. A. (3) **16**, 25–36 (1953).

Correction and continuation of Eyraud, same Ann. (3) **14**, 119–148 (1951); these Rev. **14**, 255. *F. Bagemihl.*

Bachmann, Heinz. Normalfunktionen und Hauptfolgen. Comment. Math. Helv. **28**, 9–16 (1954).

This continuation of the author's earlier paper [Vierteljschr. Naturforsch. Ges. Zürich **95**, 115–147 (1950); these Rev. **12**, 165] contains, in addition to some results on critical numbers of normal functions and principal numbers (Hauptzahlen) of arithmetic operations on ordinals, a list of seven propositions (not all new) that are equivalent, under the Zermelo-Fraenkel axiom system without the axiom of choice, to the proposition that there exists a function assigning a distinguished sequence to every limit number of the class Z_2, where $\alpha \in Z_2$ if, and only if, either $\alpha=0$, or $\alpha>0$ and α and all β with $0<\beta<\alpha$ are either successor numbers or limits of increasing sequences of type ω.

F. Bagemihl (Princeton, N. J.).

Novák, Josef. On some problems of Luzin concerning the subsets of natural numbers. Čehoslovack. Mat. Ž. **3**(78), 385–395 (1953). (Russian. English summary)

The problems of Luzin [Izvestiya Akad. Nauk. SSSR. Ser. Mat. **11**, 403–410 (1947); these Rev. **9**, 82] considered are (1), (2), (3) in the review cited, and the following problem: (4) To prove or disprove the existence of two families $\mathfrak{E}=\{E_0, E_1, \cdots, E_n, \cdots\}$ $(n<\omega)$ and

$$\mathfrak{F}=\{F_0, F_1, \cdots, F_\alpha, \cdots\} \quad (\alpha<\Omega)$$

such that $\mathfrak{E}$ and $\mathfrak{F}$ are orthogonal and strictly increasing but not separated. By means of the bicompactification $\beta(I)$ [cf. Čech. Ann. of Math. (2) **38**, 823–844 (1937)] of the set I of natural numbers, the author reduces the problems to topological ones, and gives affirmative solutions of (1) and (4) under the assumption that $2^{\aleph_0}=\aleph_1$, and of (2) and (3) under the assumption that $2^{\aleph_0}<2^{\aleph_1}$. He notes that Sierpiński [Fund. Math. **35**, 141–150 (1948); these Rev. **10**, 689] gave an affirmative solution of (3) under the hypothesis that $2^{\aleph_0}=\aleph_1$, but fails to note that Sierpiński [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **4**, 519–520 (1948); these Rev. **10**, 184] also gave an affirmative solution of (1) under the same hypothesis. *F. Bagemihl.*

Erdös, Paul. Some remarks on set theory. III. Michigan Math. J. **2**, 51–57 (1954).

[For parts I and II see Ann. of Math. (2) **44**, 643–646 (1943); Proc. Amer. Math. Soc. **1**, 127–141 (1950); these Rev. **5**, 173; **12**, 14.] Let C be the set of real numbers. I. To every $x \in C$ let there correspond a set $S(x) \subset C$ such that x non-$\in S(x)$. Two points x, y of C are called independent, if x non-$\in S(y)$ and y non-$\in S(x)$; a subset of C is said to be independent, if every pair of points of this subset is independent. The assumption that $S(x)$, for every $x \in C$, is not everywhere dense, does not imply the existence of an independent pair, even if $S(x)$ has finite measure. If, for every $x \in C$, $|S(x)|<2^{\aleph_0}$ and $S(x)$ is not everywhere dense, then there exists an independent pair but not necessarily an independent triple. If, for every $x \in C$, $|S(x)|<2^{\aleph_0}$ and x is not a two-sided limit point of $S(x)$, then there exists an infinite independent set, but not necessarily a nonenumerable one. If $S(x)$ is nowhere dense for every $x \in C$, then there exists an enumerable independent set. II. $C=A_1\cup A_2$, where, for every $z \in C$, the number of solutions of the equation $x+y=z$, with x and y both in the same A_i $(i=1, 2)$, is less than $2^{\aleph_0}$. If $C=B_1\cup B_2$, $|B_1|=|B_2|=2^{\aleph_0}$, $\mathfrak{m}<2^{\aleph_0}$, and, for every $z \in C$, the number of solutions of the equation $x+y=z$, with x and y both in B_1, is less than $\mathfrak{m}$, then, for some $z \in C$, the equation $x+y=z$ has $2^{\aleph_0}$ solutions with x and y both in B_2. *F. Bagemihl* (Princeton, N. J.).

Bagemihl, F., and Erdös, P. Intersections of prescribed power, type, or measure. Fund. Math. **41**, 57–67 (1954).

The main theorem in an earlier paper by Bagemihl [Ann. of Math. (2) **55**, 34–37 (1952); these Rev. **13**, 542] is generalized. In addition, several theorems dealing with plane point sets which intersect every straight line in a set of prescribed power, order type, or measure are given. *S. Ginsburg.*

Ginsburg, Seymour. Further results on order types and decompositions of sets. Trans. Amer. Math. Soc. **77**, 122–150 (1954).

The paper is a sequel to previous papers of the author [these Rev. **14**, 853], and deals in particular with various forms of partitions of linear sets S; thus, e.g., in §§1, 2, 3 respectively one deals with partitions in sets which are incomparable, comparable and have the A-property. Moreover, the paper contains several problems on this subject. Each S of power $2^{\aleph_0}=c$ (and satisfying $S \in (C)$) is decomposable into $\bar{\alpha}$ incomparable (and exact) sets S_ξ of power c each; here α is any ordinal $\geqq 2$ and $\leqq\omega$ (Th. 1.3 (resp., Th. 1.1)).

No S of the type η is the union of a finite number of incomparable order types (Th. 1.5). The linear continuum R is the union of $\aleph_0$ pairwise disjoint similar exact sets, of power c each; here the cardinal $\aleph_0$ can not be replaced by 2; it is an open question whether $\aleph_0$ can be here replaced by n, where $2<n<\omega$ (Th. 2.1). If S contains a system of disjoint similar sets of power c each, then S is the union of an equivalent system of disjoint similar exact sets, each of which has the A-property (Th. 2.5). For each ω-sequence of

denumerable order types $\alpha_n \neq 0$, the set of rational numbers is the union of an ω-sequence of disjoint sets S_n so that $\bar{S}_n = \alpha_n$ (Th. 2.7). Let $n < \omega$; if S contains a set D of power c such that D is the union of $n+1$ disjoint similar sets, then S is not the union of $< n+1$ disjoint sets each of which has the A-property (Th. 3.1). Each S of power c is decomposable into a family H of exact sets so that if $P \subset H$, $|P| < c$, then the set $S(P) = \bigcup x$ $(x \varepsilon P)$ has the A-property; if, moreover, $S \varepsilon (C)$, then $S(P)$ is exact (Th. 3.4). The §4 deals with the P-property and contains five theorems, the 4th of which reads: If $\{L_\xi\}_\xi$ is a sequence of linear sets, of power c each and $\bar{L}_\xi < \lambda$, then there is an exact set S of property A and such that $\bar{S} \| \bar{L}_\xi$ for each ξ. Terminology [v. loc. cit]: S, of power c, is said to have the C-property, symbolically $S \varepsilon (C)$, if each $x \varepsilon S$ is a c-condensation point of S; S is exact, if the identity is the unique similarity transformation of S into itself. *Đ. Kurepa* (Zagreb).

Ginsburg, Seymour. Fixed points of products and ordered sums of simply ordered sets. Proc. Amer. Math. Soc. **5**, 554–565 (1954).

The author examines the relationships of fixed-point properties between ordered chains and their ordered sums and alphabetically ordered products. For a given chain S let IS be the set of all the fixed (or invariant) points of S, i.e. $IS = \{x | x \varepsilon S, f(x) = x\}$, for each similarity transformation f of S into itself. If A, B are disjoint chains and if $p \varepsilon IA$, $q \varepsilon IB$, then $\{p, q\} \cap I(A+B) \neq 0$ (Le. 1). If $p \varepsilon IC$, $q \varepsilon ID$, there exists a $(r, q) \varepsilon I(C \times D)$ with $r \varepsilon IC$ such that the closed interval $[p, r]_C$ is finite (Th. 2). The following lemma is very useful. Let A, B, $A'B'$ be chains and g a similarity mapping of $A \times B$ into $A' \times B'$ so that for each $a \varepsilon A$ there are two distinct elements $u(a)$, $v(a)$ in A' so that $g(a \times B)$ contains a point of $u(a) \times B'$ and a point of $v(a) \times B'$. Then to each $a \varepsilon A$ and $q \varepsilon [u(a), v(a)]_{A'}$ corresponds a similarity transformation h of A into A' so that $h(a) \neq a$, $h(A) \subseteq \{u(a), v(a)\}$; if, moreover, $A = A'$, one can still require that $h(a) \varepsilon [u(a), v(a)]_{A'}$ (Le. 2). If B is exact, then $(p, y) \varepsilon I(A \times B)$ $(p \varepsilon IA, y \varepsilon B)$ (Th. 3). To each family F of $c = 2^{\aleph_0}$ linear sets X satisfying $tX < tR$ (R linear continuum) corresponds an exact linear set T and a one-to-one mapping $x \to B_x$ of T onto F so that if $p \varepsilon B_x$ then $(x, p) \varepsilon I(\sum_{x \varepsilon T} B_x)$ (here $\sum$ denotes the ordered summation) (Th. 4). If A has the fixed-point property, then the ordered pair (A, A) is an A-pair (Th. 6). Assume $\sim(tA \leqq tB)$; then (a) $\sim(t(C \times A) \leqq t(C \times B))$, provided that C has the fixed-point property, (b) $\sim(t(A \times C) \leqq t(B \times C))$, provided $tC < tC \cdot 2$ (Th. 7). If $tA \| tB$ and if C has the fixed-point property, then $t(C \times A) \| t(C \times B)$; if $tA \| tB$ and $tC < tC \cdot 2$, then $t(A \times C) \| t(B \times C)$ (Cor. 2 to Th. 7). Terminology: S has the fixed-point property, if each similarity transformation f of S into S has a fixed point; S is exact, if $IS = S \neq 0$. The ordered couple (X, Y) of chains is called an A-pair if for each ordered couple $(X'Y')$ of chains and each similarity f of $X \times X'$ into $Y \times Y'$ there exist two elements $p \varepsilon X$, $q \varepsilon Y$ such that $f(p \times Y) \subseteq q \times Y'$ [Ahrens, Portugaliae Math. **10**, 25–28 (1951); these Rev. **13**, 542]. Here tX means the order type of X (the author denotes tX by $|X|$); $tX \| tY$ means that tX, tY are incomparable; $\sim X$ means the negation of X. *Đ. Kurepa* (Zagreb).

Büchi, J. R. On the existence of totally heterogeneous spaces. Fund. Math. **41**, 97–102 (1954).

Call a set M, of power $2^{\aleph_0}$, of the real numbers, totally heterogeneous, if for every Borel function f of a subset $X \subseteq M$ into M, the set $\{f(x) | f(x) \neq x\}$ is of power $< 2^{\aleph_0}$. Several results on the existence of totally heterogeneous sets which satisfy additional conditions are given. A typical example is the following: If $2^{\aleph_0} = \aleph_1$, then there exists a totally heterogeneous set M which has outer measure ∞, and even stronger, the outer measure of $M \cap E$ is equal to the measure of E, for every measurable set E. Such a set is automatically of the second category. *S. Ginsburg.*

★Luzin, N. N. Lekcii ob analitičeskih množestvah i ih priloženiyah. [Lectures on analytic sets and their applications.] Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1953. 359 pp. 11.45 rubles.

Have French

A translation, with some modification, by N. K. Bari of the author's Leçons sur les ensembles analytiques et leurs applications [Gauthier-Villars, Paris, 1930]. An introduction and notes are provided by L. V. Keldyš and P. S. Novikov.

Goffman, Casper. On a theorem of Henry Blumberg. Michigan Math. J. **2**, 21–22 (1954).

A theorem of Blumberg [Trans. Amer. Math. Soc. **24**, 113–128 (1923)] asserts that for every real function $f(x)$ on $I = (0, 1)$ there is an everywhere dense set E such that the E-restriction of f is continuous. The purpose of this note is to produce a one-one mapping $y = f(x)$ of I onto itself such that, for every E which is everywhere dense in I, the E-restriction of f is not a homeomorphism between E and $f(E)$. Definition of f: $n = 1, 2, \cdots$ and for every n, $m = 0, 1, \cdots, 2^n - 1$. $I_{nm} = (m/2^n, m+1/2^n)$. The intervals $\bar{I}_{nm}$ are open to the left and closed to the right, mutually disjoint, their union is I and if $n_1 > n_2$ or if $n_1 = n_2$, $m_1 > m_2$ then $\bar{I}_{n_1 m_1}$ is to the right of $\bar{I}_{n_2 m_2}$. S_{nm} is a non-empty perfect nowhere dense subset of I_{nm} such that the sets S_{nm} are mutually disjoint. $S = \bigcup S_{nm}$; $\bar{S} = I - S$; $\bar{S}_{nm} = \bar{S} \cdot \bar{I}_{nm}$. Every S_{nm}, $\bar{S}_{nm}$ has cardinal number c. The mapping f is defined by means of arbitrary one-one mappings of S_{nm} onto $\bar{S}_{nm}$ and $\bar{S}_{nm}$ onto S_{nm} for every n and m. *Chr. Pauc.*

Cargal, Buchanan. Generalizations of continuity. Proc. Iowa Acad. Sci. **60** (1953), 477–481 (1954).

Let X, Y, be two Hausdorff spaces, f a function on X into Y, S a subset of X, and λ a property defined for subsets of X. The point $\xi \varepsilon X$ is said to be a point of λ f-approach [$k\lambda$ f-approach] by S, if for every neighborhood $M(f(\xi))$ of $f(\xi)$ and for every neighborhood $N(\xi)$ of ξ [for every neighborhood N contained in certain $N(\xi)$],

$$N(\xi)SE\{x : f(x) \varepsilon M(f(\xi))\} \quad [NSE\{x : f(x) \varepsilon M(f(\xi))\}]$$

has property λ. Using this concept, the author unifies the study of generalized continuity and results of Bledsoe [Proc. Amer. Math. Soc. **3**, 114–115 (1952); these Rev. **13**, 634] and Thielman [Proc. Iowa Acad. Sci. **59**, 338–343 (1952); these Rev. **14**, 628]. *M. Cotlar* (Mendoza).

Behnert-Smirnov, K. N. Über eine notwendige und hinreichende Bedingung der gleichgradigen Stetigkeit von Funktionenmengen. Math. Ann. **127**, 424–432 (1954).

Let A be a positive constant, let G be a domain of Euclidean n-space whose boundary points are accessible by solid angles, and let F be an even convex function of the real variable t which increases monotonely from 0 to ∞ with t. Further let $E = E(G, F, A)$ be the class of real functions $u(x)$ $(x \varepsilon G)$, to each of which there correspond n real functions $u_i(x)$ $(x \varepsilon G)$, termed generalized partial derivatives, with the following properties: (1) on every rectifiable arc in G, the line integral of the differential expression with

the coefficients $u_i(x)$ is the difference of $u(x)$ at the extremities; (2) the volume integral in G of $\sum F[u_i(x)]$ does not exceed A. The author proves that equi-continuity holds in E if and only if the integral in t of the power $-1/(n-1)$ of the derivative of F converges at $t=+\infty$. The reviewer observes that the hypotheses can be relaxed a little without affecting the arguments. The problem arose from a criterion involving higher derivatives, due to Courant, Friedrichs and H. Lewy [Math. Ann. **100**, 32–74 (1928)], and modified in the 1930's by Sobolev, Kondraschov, Petrowski and the author. *L. C. Young* (Madison, Wis.).

Temple, W. B. Stieltjes integral representation of convex functions. Duke Math. J. **21**, 527–531 (1954).

The author uses approximation by Bernstein polynomials to prove that a function f continuous and convex in an open interval containing $\langle 0, 1\rangle$ may be represented in the latter interval by the formula

$$f(x)=l(x)-\int_0^1 G_2(x,t)d\mu(t).$$

Here l is the linear function with $l(0)=f(0)$ and $l(1)=f(1)$; $G_2(x,t)=x(1-t)$ for $0<x\leqq t\leqq 1$ and $G_2(x,t)=t(1-x)$ for $0\leqq t\leqq x<1$; μ is a non-decreasing function. The same method is used to prove a similar representation theorem for convex functions of order k, $k>2$. *F. F. Bonsall.*

Have Eng.

***Halmoš [Halmos], P. Teoriya mery. [Measure theory.]** Izdat. Inostrannoĭ Literatury, Moscow, 1953. 291 pp. 16.35 rubles.

Translation of the author's "Measure theory" [Van Nostrand, New York, 1950; these Rev. **11**, 504].

Hadwiger, H., und Nef, W. Zur axiomatischen Theorie der invarianten Integration in abstrakten Räumen. Math. Z. **60**, 305–319 (1954).

The authors extend the earlier results of Hadwiger and his collaborators to integration systems of a very high degree of generality; the underlying set and the transformation group acting on it are both completely arbitrary. *P. R. Halmos* (Chicago, Ill.).

Hadwiger, H. Absolut messbare Punktmengen im euklidischen Raum. Comment. Math. Helv. **28**, 119–148 (1954).

The absolute outer content of a subset A of a finite-dimensional Euclidean space R is defined as the infimum of the quotients p/n, extended over those values of $p\geqq 0$ and $n\geqq 1$ for which there exists n translation copies of A and p translation copies of the unit cube such that the sum of the characteristic functions of the former is less than or equal to the sum of the characteristic functions of the latter. Absolute inner content is defined similarly; a set is absolutely measurable if its absolute inner content and outer content are the same. The author studies various properties of absolute measurability, and, in particular, its relation to Tarski's concept of the same name, and to von Neumann's construction of a mean value. *P. R. Halmos.*

Nef, Walter. Zerlegungsäquivalenz von Funktionen und invariante Integration. Comment. Math. Helv. **28**, 162–172 (1954).

The main point of the paper is a useful technical generalization of a result of A. Kirsch [Math. Ann. **124**, 343–363 (1952); these Rev. **14**, 28]. Where Kirsch has conditions for the existence of an "integral" on a suitable subclass of the class of all "bounded" functions, Nef presents conditions for the existence of such an integral on a suitable subclass of a prescribed subclass of the class of all bounded functions. *P. R. Halmos* (Chicago, Ill.).

McMinn, Trevor J. Measure splitting and average measurability. Proc. Amer. Math. Soc. **5**, 420–429 (1954).

S: set of points (without topology). ϕ: Carathéodory outer measure defined on all subsets of S without σ-finiteness assumption. μ: measure induced by ϕ, defined on $\mathbf{M}$. μ^*, μ_*: outer measure, inner measure induced by μ. For any set $X\subset S$, $\alpha(X)=\frac{1}{2}(\mu^*(X)+\mu_*(X))$; α is an outer measure. $\mathbf{M}^0$, $\mathbf{A}$: family of the Carathéodory μ^*-measurable, α-measurable sets. n-split ($n>1$): partition $M=\Sigma M_i$ of an $\mathbf{M}$-set M of positive finite measure into n disjointed sets M_i having each an outer measure $=\mu(M)$. It is known that $\mathbf{M}\subset\mathbf{M}^0\subset\mathbf{A}$. A. P. Morse raised the question of the equality $\mathbf{M}^0=\mathbf{A}$ and proved its validity when S is a complete metric space with countable basis and ϕ a regular outer measure. The author gives an answer in the general setting described above by the following theorems: (i) A necessary condition for $\mathbf{A}\subset\mathbf{M}^0$ is: corresponding to any 2-split $M=A+B$, there exists a 3-split of M. (ii) A sufficient condition for $\mathbf{A}\subset\mathbf{M}^0$ is: corresponding to any 2-split $M=A+B$, there exists a countable family of sets G_i covering M such that for any pair of indices i', i'', $\mu^*(AG_{i'})+\mu^*(BG_{i''})<\infty$ implies $(\mu_*(AG_{i'}\cup BG_{i''})=0$. Zorn's Lemma is used in the proofs. [Remark by the reviewer: Apparently ϕ occurs only through μ, and μ could be any measure defined on a σ-ring $\mathbf{M}\ni S$.] *Chr. Pauc* (Nantes).

Have

***Denjoy, Arnaud. Mémoire sur la dérivation et son calcul inverse.** Gauthier-Villars, Paris, 1954. vii+380 pp. 2700 fr.; \$8.00.

Reproduction by photo-offset of three papers by Denjoy [J. Math. Pures Appl. (7) **1**, 105–240 (1915); Bull. Soc. Math. France **43**, 161–248 (1916); Ann. Sci. Ecole Norm. Sup. (3) **33**, 127–222 (1916); **34**, 181–236 (1917)].

Morse, Anthony P. On intervals of prescribed lengths. Proc. Amer. Math. Soc. **5**, 407–414 (1954).

λ: sequence of non-negative real numbers λ_n with a sum $=1$. $H(\lambda)$: set of the sums of numbers of subsequences of λ. λ-file embracing a set X: sequence of open (numerical) intervals J_n such that the length of J_n is equal to λ_n for $n=1, 2, \cdots$ and the union of the J_n includes X. Theorems. (i) For any λ the closed unit interval $[0, 1]$ includes such a Lebesgue nullset A' that no λ-file embraces A'. (ii) A necessary and sufficient condition that the open unit interval $(0, 1)$ includes such a Lebesgue nullset A that no λ-file embraces A is that the Lebesgue measure of $H(\lambda)$ is equal to 0. In the proofs considerable use is made of a theorem of E. M. Beesly and A. P. Morse [Duke Math. J. **12**, 585–619 (1945), p. 608, Theorem 7.1; these Rev. **7**, 377]. *Chr. Pauc* (Nantes).

Cesari, Lamberto. Teoremi di approssimazione per superficie. Rivista Mat. Univ. Parma **4**, 255–287 (1953).

The following notations and definitions are needed to state one of the main results established in this important paper. Let T: $x=x(u,v)$, $y=y(u,v)$, $z=z(u,v)$ be a continuous mapping from the unit square $A: 0\leqq u\leqq 1, 0\leqq v\leqq 1$ into Euclidean 3-space E_3. The coordinate planes (y,z), (z,x), (x,y) in E_3 are denoted by E_{21}, E_{22}, E_{23} respectively. The continuous mapping T is thought of as a representation of a Fréchet surface. The orthogonal projections of this

surface upon the coordinate planes E_{31}, E_{23}, E_{12} are given by mappings designated by T_1, T_2, T_3. In case T is quasi-linear, the elementary area of the corresponding Fréchet surface is denoted by $a(T)$. The letter D, eventually with subscripts, will refer generically to a finite system of non-overlapping, simply connected polygonal regions in A. An equation of the form $D=D_0+D_1+\cdots+D_N$ will mean that D is made up of the mutually exclusive systems $D_0, D_1, \cdots, D_N$. The symbols $\sum$, $\sum^j$ will refer to summations extended over all the regions $q \in D$, $q \in D_j$ respectively, while $\sum_j$ will designate a summation over all the integers $j=1, \cdots, N$. If q is a simply connected polygonal region in A, then q^* is the positively oriented perimeter of q, and for $r=1, 2, 3$ one denotes by $O(p, q, T_r)$ the topological index of the point $p \in E_{2r}$ with respect to the oriented closed continuous curve $T_r(q^*)$. In the sequel, the Lebesgue area of the Fréchet surface represented by T will be finite. Then $O(p, q, T_r)$, as a function of the point $p \in E_{2r}$, is summable, and one denotes its integral over E_{2r} by $u(q, T_r)$. One denotes by $U(A, T_r)$ the least upper bound, with respect to all systems D, of the sums of the form $\sum|u(q, T_r)|$, $q \in D$. Finally, $l(C_r)$ denotes the length of the curve $T_r(A^*)$. Consider now a sequence T_n of continuous transformations from A into E_3. The symbols $l(C_{nr})$, T_{nr}, $u(q, T_{nr})$ will be used in the sense explained above in connection with T. The author establishes the following approximation theorem. Assume that (i) each T_n is quasi-linear, (ii) T_n converges uniformly to T in A, (iii) there exists a finite positive constant M such that $a(T_n), l(C_{n3})<M$. Then there exists an infinite sequence $[n]$ of positive integers n and, for every $n \in [n]$, a subdivision $D=D_0+D_1+\cdots+D_N$ of A into simply connected polygonal regions such that (1) diam $T(q)<\epsilon$ for every $q \in D_0$, diam $T(\sum^j q)<\epsilon$ for $j=1, \cdots, N$; (2) the planar measure of $T_3(\sum^0 q^*)$ is less than ϵ; (3) $U(A, T_3)-\sum^0|u(q, T_3)|<\epsilon$, $\sum_j\sum^j|u(q, T_3)|<\epsilon$; (4) $\sum^0|u(q, T_{n3})-u(q, T_3)|<\epsilon$ for every $n \in [n]$; (5) $\sum_j|\sum^j u(q, T_{n3})|<\epsilon$ for every $n \in [n]$; the polygonal regions $q \in D_0$ and the integer N are the same for every integer $n \in [n]$. Naturally, the same statement holds for the projections T_1, T_2. Beyond the intricate proof of this important approximation theorem, the paper contains statements of and additional comments on many fundamental results in surface-area theory. *T. Radó.*

Theory of Functions of Complex Variables

Pontryagin, L. S. On the zeros of certain elementary transcendental functions (supplement). Doklady Akad. Nauk SSSR (N.S.) **91**, 1279–1280 (1953). (Russian)

L'auteur indique que ses résultats de 1942 [Izvestiya Akad. Nauk SSSR. Ser. Mat. **6**, 115–134 (1942); ces Rev. **4**, 214] s'appliquent aux fonctions plus générales qu'aux quasi polynomes. Soit $h(z, t)=\sum a_{mn}z^m t^n$ un polynome en z et en t. La fonction $H(z, e^z)$ est un quasi polynome. Si $l(z)=l_0z^p+l_1z^{p-1}+\cdots+l_p$, $l_0\neq 0$, et si $\hat{H}(z)=H(z)/l(z)$ n'a pas de pôles, $\hat{H}$ est un quasi polynome généralisé. $\hat{H}$ est stable si ses zéros sont à gauche de l'axe imaginaire. h a un terme principal $a_{rs}z^r t^s$ si $a_{rs}\neq 0$, et si pour tout $a_{mn}z^m t^n$, non nul, on a $r\geqq m$, $s\geqq n$ (la combinaison $r=m$, $s=n$ étant, bien extendu exclue). Pour que $\hat{H}$ soit stable il faut que h admette un terme principal. Posons $\hat{H}(iy)=\hat{F}(y)+i\hat{G}(y)$. Pour que $\hat{H}$ soit stable il faut et il suffit que: (a) pour k entier positif assez grand $\hat{F}$ et $\hat{G}$ n'admettent pas plus de $4ks+s-p$ zéros distincts dans $[-sk\pi-\epsilon, sk\pi+\epsilon]$; (b) que les zéros de $\hat{F}(y)$ et $\hat{G}(y)$ soient entrelacés; (c) que $\hat{G}'(y)\hat{F}(y)-\hat{F}'(y)\hat{G}(y)>0$ pour tout y (cette expression ne s'annulant jamais, pour que (c) soit satisfait il suffit que cette expression soit positive pour une valeur de y). *S. Mandelbrojt.*

Goldenberg, H. Complex roots of a transcendental equation. Math. Gaz. **38**, 161–165 (1954).

A proof is given that no complex root of the equation $\coth z=(c/z)-b$, $c<1$, $b>0$, possesses a positive real part. A method for the solution of this equation is shown. *E. Frank* (Chicago, Ill.).

Ganelius, Tord. Sequences of analytic functions and their zeros. Ark. Mat. **3**, 1–50 (1954).

In the introduction to this thesis the author states a great many known results on the distribution of the zeros of the polynomials in a given sequence. It will be sufficient for the moment to just mention the names of Szegö, Erdös and Turán (equi-distribution of the arguments of the zeros of polynomials), Jentzsch and Carlson (zeros of partial sums of power series), Rosenbloom (domains containing few or many zeros relative to the degrees of the polynomials), Lindwart and Pólya (sequences of polynomials all of whose zeros lie in a sector or a half-plane). The author next states his tools, which include Jensen's theorem (for a disc), and its analogs for a finite sector, an infinite sector and a strip. He then proceeds to prove the results referred to above in a more general setting and obtains various new results besides.

The author obtains a necessary and sufficient condition for equi-distribution on (A, B) of the imaginary parts of the zeros of exponential polynomials $E_\nu(z)=\sum a_\mu^{(\nu)} \exp(\lambda_\mu^{(\nu)}z)$, $\lambda_\mu^{(\nu)}\geqq 0$ $(\nu=1, 2, \cdots)$. Applications are made to ordinary polynomials, and a comparison is made with the condition for equi-distribution which follows from the theorem of Erdös and Turán [Ann. of Math. (2) **51**, 105–119 (1950); these Rev. **11**, 431]. Next the author considers an exponential polynomial $E(z)=a_0+a_1\exp(\lambda_1 z)+\cdots+a_n\exp(\lambda_n z)$, $a_0a_n\neq 0$, $\lambda_{\mu+1}-\lambda_\mu\geqq\gamma>0$ $(\lambda_0=0)$. He obtains the following estimate for the number $N(\delta, T)$ of zeros of $E(z)$ in $|\mathrm{Re}\, z|\leqq\delta$, $|\mathrm{Im}\, z|\leqq T$:

$$-3\left(\varphi^{\frac{1}{2}}+3\frac{\varphi}{\delta}\right)\leqq\frac{N(\delta, T)}{\lambda_n}-\frac{T}{\pi}\leqq 6(e^{1/\delta}\varphi^{\frac{1}{2}}+\varphi),$$

where

$$\varphi=\left(T+\frac{\pi}{\gamma}\right)\frac{1}{\lambda_n}\log\frac{|a_0|+|a_1|+\cdots+|a_n|}{|a_0a_n|^{\frac{1}{2}}}.$$

This estimate contains the theorem of Erdös and Turán for ordinary polynomials as the special case $\lambda_n=n$, $\delta\to\infty$. A simple direct proof of the theorem of Erdös and Turán (with an improved constant) is based on the following theorem. Suppose that $f(z)=u+iv$, $f(0)=0$, is regular for $|z|<1$, and suppose that $u<H$, $\partial v/\partial\theta<K$ for $|z|<1$. Then for every α and β, and $\rho<1$, $v(\rho e^{i\beta})-v(\rho e^{i\alpha})>-13(HK)^{\frac{1}{2}}$ (and hence $|v(z)|<13(HK)^{\frac{1}{2}}$ for $|z|<1$). Taking $f(z)=\log P(z)$, one immediately obtains the theorem of Erdös and Turán for the crucial case where all zeros of the polynomial $P(z)$ lie on $|z|=1$.

Two results are proved which exhibit domains with many zeros. The first one contains Jentzsch's theorem on the clustering of zeros of partial sums as a very special case. Let $f_\nu(z)$, $\nu=1, 2, \cdots$, be regular in the closure of a bounded domain Ω, let $\lambda_\nu=\sup_\Omega\log|f_\nu(z)|\to\infty$, and let $|f_\nu(z_0)|\geqq m>0$ where $z_0 \in \Omega$. Let $\lim(1/\lambda_\nu)\log^+|f_\nu(z)|=0$ for $z=z_0$ and let E be the largest domain containing z_0 such that this relation holds for every $z \in E$. Let ζ be a boundary point of E con-

tained in Ω, and let V denote an arbitrary neighborhood of ζ. Then the number of zeros of $f_\nu(z)$ in V is not $o(\lambda_\nu)$. [The reviewer found a slip in the proof on p. 36. Since $r=r_\nu$, the proof given is not valid. One may, however, consider the mean value of $\log^+|h_\nu|$ over the annulus $a\leqq r\leqq 2a$; this mean value tends to zero and hence $\log^+|h|=0$.] The second theorem contains Carlson's result on the zeros of partial sums of entire functions of order ρ, $0<\rho<\infty$ [Ark. Mat. Astr. Fys. **35A**, no. 14 (1948); these Rev. **10**, 27]. Let the polynomials $P_\nu(z)$ of degree n_ν satisfy $n_\nu\to\infty$, $P_\nu(0)=1$. Denote $\max_{|z|=r}|P_\nu(z)|$ by $M_\nu(r)$. Suppose there exist numbers r_1, r_2 $(0<r_1<r_2)$ such that

$$\lim n_\nu^{-1}\log M_\nu(r_1)=0,\quad \limsup n_\nu^{-1}\log M_\nu(r_2)=\eta>0.$$

Then there exists a subsequence $P_{\nu_i}(z)$ with the following property. For every domain S containing the origin and extending to infinity there is a number $k(S)>0$ such that the number of zeros of $P_{\nu_i}(z)$ in S is at least $k(S)n_{\nu_i}$. The reviewer remarks that the above theorems are closely related to forthcoming results of Rosenbloom [to appear in the Proceedings of the Conference on functions of a complex variable, held at the University of Michigan, summer 1953], and of A. F. Kay [Distribution of zeros of sequences of polynomials of unbounded degree, to appear].

The author finally poses the following problem. Let the zeros of the polynomials of a sequence all lie in a set R. For what sets R does the uniform convergence of the sequence in $|z|<1$ (to a limit function $\not\equiv 0$) imply its uniform convergence in every bounded domain? One result of Lindwart and Pólya requires that the half-plane $\operatorname{Re} z\geqq 0$ be free of zeros [Rend. Circ. Mat. Palermo **37**, 297–304 (1914)]. The author weakens this requirement as follows: $\operatorname{Re} z\geqq a$ should be free of zeros for some $a>0$, together with a domain which has points in common both with the unit disc and with the half-plane $\operatorname{Re} z\geqq a$. The reviewer remarks that in his terminology, R must be non-regular [Duke Math. J. **18**, 573–592 (1951); these Rev. **13**, 222]. Perhaps R could be any non-regular set whose complement is connected and contains part of the unit disc. *J. Korevaar.*

Ricci, Giovanni. Emisimmetria di tratti e teorema di Vivanti-Pringsheim-Hadamard-Fabry relativo ai punti critici. Boll. Un. Mat. Ital. (3) **9**, 126–135 (1954).

Let $f(z)=\sum a_nz^n$ have radius of convergence 1. Sufficient conditions that $z=1$ should be a singular point for this branch of $f(z)$ are (Vivanti-Pringsheim) that sequences (n_k), (γ_k) should exist such that (a) $\lim_{k\to\infty}|\operatorname{Re}(a_{n_k}e^{-i\gamma_k})|^{1/n_k}=1$, (b) $\operatorname{Re}(a_me^{-i\gamma_k})>0$ for all a_m in the range

$$\text{(c)}\qquad (1-\theta)n_k\leqq m\leqq(1+\theta)n_k$$

for some $0<\theta<1$. Sufficient conditions that $|z|=1$ should be a natural boundary for $f(z)$ are (Hadamard-Fabry) that (a) should be satisfied and that all $a_m=0$ in (c) except a_{n_k}. The author is concerned with sharpening these conditions. This he does in two ways: first by using symmetric relations involving both a_{n_k-u} and a_{n_k+u} over half the range and secondly by relaxing the conditions on the a_n over a range more restricted than (c), as has been done in a different way by Lösch [Math. Z. **32**, 415–421 (1930)] and Claus [ibid. **49**, 161–191 (1943); these Rev. **5**, 176]. The results are too lengthy to be given here but it is interesting to note that the extent of the restricted range is also the lower bound of the size of the gaps arising when the Hadamard order at any point on the natural boundary $|z|=1$ is the same as that on the whole circumference [E. Fabry, C. R. Acad. Sci. Paris **151**, 922–925 (1910)]. *R. Wilson* (Swansea).

Berg, Lothar. Über Potenzreihenteilsummen beschränkter Funktionen. Math. Nachr. **11**, 213–218 (1954).

Let $f(z)=\sum a_\nu z^\nu$ be regular and bounded in $|z|<1$. Let $b_n=s_n/\log n$, where $s_n=a_0+a_1+\cdots+a_n$. It was shown by E. Landau that $b_n=O(1)$ and by H. Bohr that b_n need not be $o(1)$. The author considers the case that $b_n\neq o(1)$ and obtains the following properties of the sequence $\{b_n\}$ under this assumption: (1) $\{b_n\}$ diverges; (2) $\lim(b_n-b_{n-1})=0$; (3) the set of limit points of $\{b_n\}$ is a connected continuum, and at least one limit point lies on the real axis and one on the imaginary axis; (4) $\sum|b_n-b_{n-1}|^2<\infty$; (5) $\liminf n^{1/2}|b_n-b_{n-1}|=0$; (6) $\limsup n|b_n-b_{n-1}|=\infty$. Finally the author considers a special function with the above-mentioned properties for which $b_n=\exp(in^\alpha)$ with $0<\alpha<\frac{1}{2}$. Since in this case $|b_n-b_{n-1}|\sim\alpha n^{\alpha-1}$, the exponents of n in (5) and (6) above cannot be improved. It may be noted that the second part of property (3) above holds not only for the two axes, but for every line through the origin. *F. Herzog* (East Lansing, Mich.).

Hiong, King-Lai. La normalité d'une famille de fonctions holomorphes en liaison avec le défaut d'une valeur de leurs dérivées. C. R. Acad. Sci. Paris **238**, 2279–2281 (1954).

Let $f(z)$ be regular in $|z|<1$, let $f(z)$ have at most p zeros, and let $f'(z)$ have 1 as a defective value. The author announces an inequality for $\log|f(z)|$ [cf. his related work in same C. R. **236**, 1628–1630 (1953); Ann. Sci. Ecole Norm. Sup. (3) **70**, 149–180 (1953); these Rev. **14**, 859; **15**, 412]. *J. Korevaar* (Madison, Wis.).

Reich, Edgar. An alternative proof of a theorem of Beckenbach. Proc. Amer. Math. Soc. **5**, 578–579 (1954).

If $f(z)$ is regular for $|z|<1$ and

$$I(\rho,\theta)=\int_0^\rho|f(re^{i\theta})|\,dr\leqq 1$$

for $0\leqq\rho<1$, $0\leqq\theta<2\pi$, then $I(\rho,\theta)\leqq\rho$. Equality for any (ρ_0,θ_0) with $\rho_0>0$ implies $f(z)=e^{i\alpha}$, α real. The author gives a short purely complex-variable proof of this analogue of Schwarz's lemma. The original proof, due to Beckenbach [Bull. Amer. Math. Soc. **44**, 698–707 (1938)] relies on the theory of subharmonic functions. *W. Rudin.*

Jenkins, James A. On Bieberbach-Eilenberg functions. Trans. Amer. Math. Soc. **76**, 389–396 (1954).

Let C be the class of functions $f(z)=a_1z+a_2z^2+\cdots$ regular in $|z|<1$, and satisfying $f(z_1)f(z_2)\neq 1$, $|z_1|<1$, $|z_2|<1$. Rogosinski [J. London Math. Soc. **14**, 4–11 (1939)] showed that $f(z)\in C$ is subordinate to a univalent function $\bar f(z)=\bar a_1z+\bar a_2z^2+\cdots\in C$. Bieberbach [Math. Ann. **77**, 153–172 (1916)] had shown that for $\bar f(z)$, $|\bar a_1|\leqq 1$ holds and hence also $|a_1|\leqq 1$ for $f(z)\in C$ by subordination. The author now shows the following. I. $|\bar f(re^{i\theta})|\leqq r(1-r^2)^{-1/2}$, with equality only for $\bar f_\tau(ze^{i\theta})=\mp(1-r^2)^{1/2}ze^{i\theta}/(1+irze^{i\theta})$. II. He further obtains the extremal functions giving the exact upper and lower bounds for $\bar f(re^{i\theta})$, when $\bar f(z)\in C$, is univalent, and $|\bar a_1|=c$, $0<c<1$ is given.

The inequality I remains true for general $f(z)\in C$ by subordination. However, the author's claim that the upper bound result in II remains true for general $f(z)\in C$ is not true, at any rate for small c. For example, $Z^2\in C$, but there is no univalent $\bar f(z)\in C$ satisfying $a_1=0$. *W. K. Hayman* (Exeter).

Ohtsuka, Makoto. Note on functions bounded and analytic in the unit circle. Proc. Amer. Math. Soc. 5, 533–535 (1954).

The author constructs a function $f(z)$ with the following properties. $f(z)$ is regular and $|f(z)|<1$ in $|z|<1$ and the radial limits $f(e^{i\theta})$ satisfy $|f(e^{i\theta})|=1$, except on a set of θ of linear measure zero. Given any complex α, with $|\alpha|<1$, there exist continuum-many distinct θ, $0\leqq\theta<2\pi$, such that $f(e^{i\theta})$ exists and equals α. He obtains his result by constructing a Riemann surface R over $|w|<1$ in which the harmonic measure of the part of the boundary over $|w|=1$ is one, while at the same time there exist continuum-many non-homotopic paths in R, which approach the ideal boundary, and whose projections onto the w-plane approach any preassigned α. The function $f(z)$ maps $|z|<1$ onto the infinite covering surface of R. *W. K. Hayman* (Exeter).

Polak, A. I. On a property of locally univalent functions of a complex variable. Doklady Akad. Nauk SSSR (N.S.) 96, 241–243 (1954). (Russian)

Let Z be a bounded closed locally connected set in the z-plane, and let $f(z)$ be a continuous locally univalent function defined on Z ($f(z)$ need not be regular). The author proves that in the set $W=f(Z)$ there is a closed set, nowhere dense in W, which divides W into a finite or countable system of sets $\{G_i\}$, each open and connected in W and each having the property that the number of values which belong to the inverse function $z=\varphi(w)$ is constant throughout the domain. *A. W. Goodman* (Lexington, Ky.).

Townes, S. B. A theorem on schlicht functions. Proc. Amer. Math. Soc. 5, 585–588 (1954).

I is a complex integral domain of characteristic zero for which 0 is the only integer of absolute value less than 1. It is proved that the functions $f(z)=z+a_2z^2+\cdots$ with $a_i\in I$ that are schlicht in $|z|<1$ are exactly the functions z and $z/[h(z)]^{1/n}$, where $h(z)=1+c_1z+\cdots+c_mz^m$ with $c_i\in I$ is a polynomial of degree $m\leqq 2n$ all of whose roots lie on $|z|=1$ [cf. B. Friedman, Duke Math. J. 13, 171–177 (1946), where the a_i are rational integrals; these Rev. 8, 22].

W. W. Rogosinski (Newcastle-upon-Tyne).

Wigner, E. P., and v. Neumann, J. Significance of Loewner's theorem in the quantum theory of collisions. Ann. of Math. (2) 59, 418–433 (1954).

The quantum theory of collision systems leads to functions $R(z)$ with the properties (P): $R(z)$ is defined, real-valued and differentiable for all $z>0$, except for isolated singular points Z_ν. The matrix

$$\kappa_{ii}=R'(z_i),\quad \kappa_{ik}=\frac{R(z_i)-R(z_k)}{z_i-z_k}\quad (i,k=1,\cdots,n)$$

is positive semi-definite for every n and non-singular points z_i. It is shown that $R(z)$ is either continuous at a singularity Z_ν or tends to infinity on approach from either side. In the first case Z_ν is called an apparent singularity (a.s.), in the second case a real singularity (r.s.). $R(z)$ can be defined at an a.s. to be continuously differentiable there and to still satisfy the postulate (P). $S(z)=-R(z)^{-1}$ satisfies also the postulate (P) and if Z_ν is a r.s. for $R(z)$, it is an a.s. for $S(z)$ such that $S(Z_\nu)$ and $S'(Z_\nu)$ are well defined.

$$R_\nu(z)=R(z)-[S'(z_\nu)(Z_\nu-z)]^{-1}$$

satisfies again (P) and has Z_ν as an a.s. only. This fact allows a development of $R(z)$ into a continued fraction with rational approximants and, hence, its analytic continuation into the complex plane. It is shown that $R(z)$ is meromorphic in the entire z-plane, slit along the negative axis; that is, has only simple poles, situated on the positive axis and with negative residues.

Wigner has studied previously so-called R-functions [e.g., Amer. Math. Monthly 59, 669–683 (1952); these Rev. 14, 460] which are meromorphic in the entire z-plane and have a positive (negative) part in the upper (lower) half-plane, respectively. It is now shown that these R-functions satisfy the postulate (P) and that, conversely, each function $R(z)$ is a limit of such R-functions.

The close relation between the above theory and Löwner's theory of monotonic matrix functions [Math. Z. 38, 177–216 (1934)] is pointed out. *M. Schiffer.*

Evgrafov, M. A. On the construction and uniqueness of an entire function $F(z)$ for given values $F^{(n)}(n^2)$. Izvestiya Akad. Nauk SSSR. Ser. Mat. 18, 201–206 (1954). (Russian)

By applying the theory developed in a previous paper [same Izvestiya 17, 421–460 (1953); these Rev. 15, 515] the author proves the following theorems. (1) If

$$F(z)=\sum_{n=0}^{\infty}\frac{a_n}{(2n)!}z^n,\quad f(z)=\sum_{n=0}^{\infty}a_nz^{-n-1}$$

and if all singular parts of $f(z)$ lie in a domain Δ of univalence of $g(z)=((1+z)^{1/2}-1)\exp((1+z)^{1/2}-1)$ containing $z=0$, then $F^{(n)}(n^2)=0$ $(n=0,1,\cdots)$ implies $F(z)\equiv 0$. (2) If the singular points of $f(z)$ lie in a domain D such that $0\in D$, $|g(z)|<1$ in D, then $F(z)=\sum_{n=0}^{\infty}F^{(n)}(n^2)p_n(z)$, where the p_n are interpolation polynomials. (3) The domains D and Δ in these results cannot be replaced by larger ones. Similar results can be obtained for $F^{(n)}(n^k)$ in place of $F^{(n)}(n^2)$ (k positive integer). *W. H. J. Fuchs* (Ithaca, N. Y.).

Kuroda, Tadashi. On the classification of symmetric Fuchsian groups of genus zero. Proc. Japan Acad. 29, 431–434 (1953).

The author gives sufficient conditions for the Riemann surface associated with a symmetric Fuchsian group of genus zero to belong to O_G or to O_{AD}. The conditions are expressed in terms of the divergence of integrals whose integrands reflect the behavior of a fundamental polygon of the group. The author remarks that his theorems are similar to results of Laasonen [Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. 11 (1942); these Rev. 8, 24] and Sario [ibid. no. 50 (1948); these Rev. 10, 365]. *M. Heins.*

Witt, Ernst. Über die Konstruktion von Fundamentalbereichen. Ann. Mat. Pura Appl. (4) 36, 215–221 (1954).

Siegel has proved the following theorem [Amer. J. Math. 65, 1–86 (1943); these Rev. 4, 242]: Every discontinuous subgroup Δ of the symplectic group of degree n has a fundamental region F which is a star bounded by analytic surfaces such that every compact domain in H is covered by a finite number of images of F under Δ. Here H is a space introduced by Siegel which reduces to the upper half-plane when $n=1$ [loc. cit.]. In the present paper, the author establishes quite simply a more general theorem about a locally Euclidean Riemannian space Z and a group G acting on Z, about which it is assumed only that (1) any two points in Z are connected by a unique geodesic, (2) G preserves distance in Z, (3) for each fixed $Z_0\in Z$, the set of $g\in G$ such that gZ_0 lies in a pre-

assigned compact set is itself compact. The conclusion is that every discontinuous subgroup J of G possesses a fundamental region with essentially the properties stated in Siegel's theorem.

In his proof Siegel used a representation of the symplectic group by matrices. Since this is not available for a general group G, the author has recourse to a classical existence proof [cf. Weyl, Die Idee der Riemannschen Fläche, Teubner, Leipzig-Berlin, 1913, pp. 154, 155]; namely, he considers for a fixed point $Z_0 \in Z$ and its images Z_h, $h=0, 1, \cdots$, under J the set of points P_k which are as near to Z_k as they are to Z_h, $h \neq k$. P_k is then shown to be a fundamental region with the desired properties.

The author concludes by verifying that the symplectic group G and the associated Siegel space H satisfy (1)–(3), and Siegel's theorem follows. In this verification he uses facts about G and H proved by Siegel (and Maas) from the representation of G by matrices. *J. Lehner.*

Hersch, Joseph. A propos d'un problème de variation de R. Nevanlinna. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. **168**, 7 pp. (1954).

A quadrilateral $Q\{\alpha', \beta', \alpha'', \beta''\}$ is defined as a Jordan domain whose boundary is divided into four arcs α', β', α'', β'' in that order. If Q is mapped 1:1 conformally onto a rectangle so that α', α''; β', β'' correspond to pairs of sides of lengths a, b respectively, then the modulus $\mu=\mu_{\beta'\beta''}$ of Q is a/b. The author obtains the upper bound for μ, given the harmonic measures of each of β', β'' at some point p of Q. In case Q is a rectangle, equality takes place if and only if p lies on the straight line segment joining the midpoints of β', β'' in Q. *W. K. Hayman* (Exeter).

Juve, Yrjö. Über gewisse Verzerrungseigenschaften konformer und quasikonformer Abbildungen. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. **174**, 40 pp. (1954).

This paper is concerned with distortion properties of quasi-conformal mappings. The topics treated include: (1) the existence of a Koebe constant for quasi-conformal maps, (2) an extension of the Ahlfors distortion theorem to quasi-conformal maps, (3) some refinements of the Denjoy-Carleman-Ahlfors theorem. The Koebe theorem of the author is: Let w denote a univalent quasi-conformal map of $|z|<1$ into the finite plane which is normalized to satisfy $w(0)=0$ and $\lim_{z\to0}|w|\,|z|^{-1/[d(0)]}=1$, where $d(z)$ denotes the dilatation quotient of w at z. Let W denote the class of such w which satisfy the condition

$$\int_0^1\left[\frac{1}{d(0)}-\left(\frac{1}{2\pi}\int_0^{2\pi} d(re^{i\varphi})d\varphi\right)^{-1}\right]\frac{dr}{r}\le M.$$

Then W possesses the Koebe constant $\frac{1}{4}e^{-M}$. In his investigations on the Denjoy-Carleman-Ahlfors theorem the author obtains an inequality yielding the refinement of Ahlfors which involves the Drehungsindex. For certain types of configurations the author obtains stronger estimates. Specific examples are studied. *M. Heins.*

Kuramochi, Zenjiro. On covering property of abstract Riemann surfaces. Osaka Math. J. **6**, 93–103 (1954).

A sufficient condition is given for every non-constant meromorphic function f on a Riemann surface R to have the property that for each disk Δ of the extended plane on each component of $f^{-1}(\Delta)$, f takes all values of Δ save a set which is the boundary of a plane region of class O_{AB}. The condition is the existence of a conformal metric on R which satisfies a condition of the type given by Pfluger [C. R. Acad. Sci. Paris **230**, 166–168 (1950); these Rev. **11**, 342], but slightly stronger. It is also shown that if R has finite genus and belongs to O_{AB} (O_{AD}) and the valence of f on a component of $f^{-1}(\Delta)$ is finite, then the restriction of f to this component attains all values of Δ save for a null set in the above sense. An example of a planar Riemann surface of class O_{AB} which does not have the Gross property is constructed. *M. Heins* (Providence, R. I.).

Hirzebruch, Friedrich. Über vierdimensionale Riemannsche Flächen mehrdeutiger analytischer Funktionen von zwei komplexen Veränderlichen. Math. Ann. **126**, 1–22 (1953).

Soit M une variété analytique complexe, de dimension complexe 2, et soit f une fonction algébroïde définie sur M. En chaque point P de M, la fonction multiforme f définit un ou plusieurs éléments de fonction algébroïde A_P. Un tel élément est le quotient d'un germe de fonction multiforme w par un germe de fonction holomorphe g; le germe w est défini par l'équation $\pi=0$, où π est un polynôme distingué irréductible en $(w-w_0)$ dont les coefficients sont des fonctions holomorphes des coordonnées locales en P et où w_0 désigne la valeur de w en P; enfin, le discriminant $D_\pi(P)$ du polynôme π n'est pas identiquement nul. L'ensemble des A_P est muni d'une topologie séparée, on l'appelle domaine (Bereich) de Riemann B de f et on désigne par ψ la projection de B sur M qui applique A_P sur P. Un point A_P de B est dit uniformisable s'il existe un voisinage U de A_P dans B, un voisinage $\tilde{U}$ de l'origine de l'espace numérique complexe de dimension 2, et une application topologique κ de $\tilde{U}$ sur U telle que $\psi\kappa$ soit une application analytique de $\tilde{U}$ sur $\psi(U)$; A_P est un point de ramification si $D_\pi=0$ en P. Si A_P n'est pas uniformisable, $\psi(A_P)$ est un point singulier de la surface de ramification de f (définie localement par $D_\pi=0$); les points non uniformisables de B sont donc isolés et l'ensemble des points uniformisables de B est une variété analytique complexe. La fonction f sur M définit une fonction uniforme sur B notée encore f. L'auteur construit une variété analytique complexe $\tilde{M}$ bien déterminée, appelée surface de Riemann (à 4 dimensions) de f et une application continue τ de $\tilde{M}$ sur B telles que: (1) $\psi\tau$ soit une application analytique de $\tilde{M}$ sur $\psi(B)=M$; (2) la fonction $\tilde{f}=f\tau$ soit méromorphe et sans point d'indétermination; (3) si B^* est la variété complémentaire de l'ensemble des points non uniformisables de B et des points d'indétermination de f, alors, pour tout point $Q \in B$, Q non-$\in B^*$, l'ensemble $\tau^{-1}(Q)$ soit la réunion S_Q d'un nombre fini de surfaces analytiques irréductibles compactes immergées dans $\tilde{M}$; (4) τ soit une application analytique biunivoque de $\tilde{M}-\bigcup S_Q$ sur B^*.

La technique repose sur le procédé local d'adjonction de 2-sphères (ou droites projectives complexes) à une variété analytique complexe [processus σ de H. Hopf, Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **10**, 169–182 (1951); ces Rev. **13**, 861]. Une application continue t d'un espace N dans un espace M, induit une application t^* des fonctions définies sur M dans les fonctions définies sur N. On applique le processus σ aux points non uniformisables de la surface de ramification de f qui ne sont pas des points doubles, pour obtenir une variété analytique complexe projetée sur M par une application analytique t, telle que t^*f soit méromorphe sur $t^{-1}M$ et ait une surface de ramification dont les points non uniformisables sont tous des points doubles. De plus, on construit $t^{-1}M$ de manière à adjoindre à M le moins de sphères possible. Soit $t^*B(f)$ le domaine de

Riemann de t^*f. Les éléments de fonction algébroïde déterminés par t^*f, aux points non uniformisables de $t^*B(f)$ sont définis par des polynômes distingués π du type $w^n - z_1 z_2^{n-q}$ ($1 \leqq q < n$; n, q premiers entre eux). En chaque point non uniformisable, l'auteur adjoint des 2-sphères de façon canonique et obtient une variété complexe $\hat{M}(f)$ dont la structure dépend uniquement de f. L'application induite par la projection de $\hat{M}(f)$ sur $t^*B(f)$ transforme f en une fonction méromorphe $\hat{f}$ sur $\hat{M}(f)$. Aux points d'indétermination de $\hat{f}$, on adjoint des sphères (encore par un procédé canonique), de manière à obtenir la variété complexe $\tilde{M}$ dans laquelle l'image de f est sans point d'indétermination. Technique et résultats généralisent ceux de la géométrie algébrique [H. W. E. Jung, J. Reine Angew. Math. **133**, 289–314 (1908)]. La construction est généralisée au cas où B est un domaine de Riemann abstrait, c'est-à-dire un domaine (Gebiet) de Riemann au sens de Behnke-Stein [Math. Ann. **124**, 1–16 (1951); ces Rev. **13**, 644], dont chaque point possède un voisinage analytiquement homéomorphe au revêtement ramifié d'une boule $|z_1|^2 + |z_2|^2 < \epsilon$ engendré par un élément de fonction algébroïde au point $(0, 0)$.

P. Dolbeault (Paris).

Bremermann, Hans J. Die Holomorphiehüllen der Tuben- und Halbtubengebiete. Math. Ann. 127, 406–423 (1954).

Let z and $w = u + iv$ be complex variables. A domain D defined by a condition $(z; u) \in B$, where B is a domain in a 3-dimensional space R_3 or in a Riemann manifold over R_3, is called a semitubular domain (Halbtubengebiet). The domain D is a regularity domain if and only if the domain B is defined by relations of the form $z \in Z$, $S_1(z) < u < S_2(z)$, where Z is a domain in R_2 or on a Riemann surface, while $S_2(z)$ is superharmonic in Z and $S_1(z)$ is subharmonic in Z. The author proves that this condition is necessary. The sufficiency of the condition was proved in the author's thesis [Wilhelms-Univ. zu Münster, 1951; these Rev. **15**, 25]. The author proves further that every semitubular domain has an analytic completion which is a semitubular regularity domain. The proof depends on the following "continuity theorem": Let $E(t)$, $0 < t < 1$, denote a family of parallel analytic planes intersecting a fixed plane F in a point which depends continuously on t, and converging to a limit plane E_0. Let $G(t)$ be a domain in $E(t)$ converging to a limit domain G_0. If a function $f(z; w)$ is analytic at every point of every $G(t)$, then it is either analytic at every point of G_0 or singular at every point of G_0. *H. Tornehave.*

Hörmander, Lars. On a theorem of Grace. Math. Scand. 2, 55–64 (1954).

The theorem of the title gives relations between the roots of apolar complex polynomials [see Marden, Geometry of the zeros . . . , Math. Surveys, no. 3, Amer. Math. Soc., New York, 1949, pp. 45–47; these Rev. **11**, 101]. The author generalizes this theorem in several directions; abstract homogeneous polynomials on a vector space E with values in a vector space G (both over an arbitrary algebraically closed field of characteristic zero) are considered. These generalizations are applied principally to broaden and obtain simpler proofs of some known inequalities in the theory of functions of several complex variables. *M. Henriksen.*

Caccioppoli, Renato. Funzioni pseudo-analitiche e rappresentazioni pseudo-conformi delle superficie riemanniane. Ricerche Mat. 2, 104–127 (1953).

Let A and S be two abstract Riemann surfaces, on each of which a Riemannian metric is defined and let $p = f(q)$ be a mapping of points $q \in A$ into points $p \in S$. With $|q, q'|$ and $|p, p'|$ denoting the distances between pairs of points measured in the metric of the corresponding surfaces, define the functions

$$\varphi(q) = \liminf_{q \to q'} \frac{|f(q), f(q')|}{|q, q'|} \text{ and } \phi(q) = \limsup_{q \to q'} \frac{|f(q), f(q')|}{|q, q'|}.$$

If almost everywhere on A $\varphi(q)/\phi(q) \geqq \mu$, the author calls the mapping pseudo-analytic with the parameter μ. Various criteria are given that a family of such functions be normal. The most important is the following. Let $\psi(t)$ be a continuous function which increases with t monotonically from 0 to ∞ and such that $\int_t^\infty \psi^{-2} dt$ is convergent for $t > 0$. Suppose that on S the isoperimetric inequality $l > \psi(a)$ is satisfied for the area a of every subdomain and the length l of its boundary. If A lies over the complex z-plane, each family of pseudo-analytic functions with parameter μ and mapping into S will be normal.

The results are applied to prove various theorems of the Picard type and classical results of Schottky, Landau, and Montel are generalized. The basic idea of proof is the use of universal covering surfaces on which the above isoperimetric inequality can be asserted. The metric-topological character of the Picard-type theorems is clearly exhibited. The methods are simple and elegant and of interest also in the case that A and S are in the complex z-plane (quasiconformal case). The paper represents an amplification and extension of four previous notes by the author [C. R. Acad. Sci. Paris **235**, 116–118, 228–239 (1952); Atti. Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **13**, 197–204, 321–329 (1952); these Rev. **14**, 364; **15**, 117]. *M. Schiffer.*

Theory of Series

Krzyż, Jan. On monotonity-preserving transformations. Ann. Univ. Mariae Curie-Skłodowska. Sect. A. 6 (1952), 91–111 (1954). (Polish and Russian summaries)

Krzyż, Jan. A correction to "On monotonity-preserving transformations" (Ann. UMCS, vol. VI (1952), pp. 91–111). Ann. Univ. Mariae Curie-Skłodowska. Sect. A. 6 (1952), unpaged (1954).

Real functions $\phi_1(t), \phi_2(t), \cdots$ of a real variable t determine a transformation $\Phi(t) = \sum_{k=1}^\infty \phi_k(t) x_k$ which is said to preserve monotonicity if $\Phi(t)$ is monotone increasing (or decreasing) whenever x_k is a real monotone increasing (or decreasing) sequence belonging to the class X of sequences for which the series for $\Phi(t)$ converges for each t. Regularity of the transformation is not assumed. In case each monotone sequence in X converges to 0, the transformation preserves monotonicity if and only if $\phi_1(t) + \phi_2(t) + \cdots + \phi_r(t)$ is, for each $r = 1, 2, \cdots$, a monotone decreasing function of t. In case X contains at least one monotone sequence which converges to a number different from zero, the transformation preserves monotonicity if and only if the above condition holds and the series $\sum \phi_k(t)$ converges to a number independent of t. A slight modification of the conditions gives conditions that $\Phi(t)$ be increasing (or decreasing) whenever x_k is an increasing (or decreasing) sequence in X. Related results involving matrix and kernel transformations are given. *R. P. Agnew* (Ithaca, N. Y.).

Meyer-König, W., und Zeller, K. Über das Taylorsche Summierungsverfahren. Math. Z. 60, 348–352 (1954).

The transformation of a series $\mathfrak{a}=\sum a_k$ into the series $\mathfrak{b}=\sum b_n$

$$\mathfrak{b}=T_\alpha\mathfrak{a}\equiv b_n=(1-\alpha)^n\sum_{k=n}^{\infty}\binom{k}{n}\alpha^{k-n}a_k,\quad 0<\alpha<1,$$

defines a regular method of summation. Translation of indices in $\mathfrak{a}$ does not change its T_α summability unless $0<\alpha\leqq\frac{1}{2}$ and $\sum a_k z^k$ is singular at $z=\alpha$. The system $\mathfrak{b}=T_\alpha\mathfrak{a}$ has an infinity of linearly independent solutions $\mathfrak{a}$ for each $\mathfrak{b}$. Some related methods are considered. *G. G. Lorentz.*

Amir (Jakimovski), Amnon. Some relations between the methods of summability of Abel, Borel, Cesàro, Hölder and Hausdorff. J. Analyse Math. 3, 346–381 (1954).

Let σ_n^α be the (C, α) transform of a sequence s_n. The main content of the paper consists in Tauberian theorems for different methods of summation A of the following type. (I) If s_n is A-summable and satisfies the Tauberian condition (*) $\sigma_n^\alpha-\sigma_n^\beta=o(1)$ (or $O(1)$), then s_n is summable by a (possibly weaker) method B. We quote as examples: (1) If s_n is Abel summable, $\alpha>-1$, then the necessary and sufficient condition for s_n to be (C, α) summable is that for some $\beta>\alpha$, $\sigma_n^\alpha-\sigma_n^\beta\rightarrow 0$. (2) Let s_n be Borel summable, $\beta>\alpha>-1$ and $\sigma_n^\alpha-\sigma_n^\beta=O(1)$. Then s_n is $(C, \alpha+)$ summable (i.e. $(C, \alpha+\epsilon)$ summable for each $\epsilon>0$). In particular, for $\alpha=0$, $\beta=1$, (*) reduces to the Kronecker condition $n^{-1}(a_1+2a_2+\cdots+na_n)\rightarrow 0$.

The author says that s_n is $I(\alpha, \beta)$ summable to s $(\beta>\alpha>-1)$ if

$$(**)\quad \sigma_0^\alpha+\sum_{n=1}^{\infty}\frac{\sigma_n^\alpha-\sigma_n^\beta}{[I(\beta)-I(\alpha)]n}=s,\quad I(\alpha)=\int_0^1\frac{1-u^\alpha}{1-u}du.$$

$I(\alpha, \beta)$ is a regular Hausdorff method of summation generated by the moment sequence $\mu_0=1$,

$$\mu_n=n^{-1}[I(\beta)-I(\alpha)]^{-1}\left\{\frac{1}{\binom{n+\alpha}{n}}-\frac{1}{\binom{n+\beta}{n}}\right\},\quad n=1, 2, \cdots.$$

Using Hausdorff's theorems on comparison of Hausdorff methods of summation, the author shows that $I(\alpha, \beta)$ is equivalent to $(C, \alpha+1)$. The method of proof of Theorem (I) is then as follows. It is shown that A-summability of s_n implies the $AI(\alpha, \beta)$ summability, i.e. the A-summability of the series (**). Using a condition of type (*) and a standard Tauberian theorem for the method A, convergence of the series (**) is deduced, and this means the $(C, \alpha+1)$ summability of s_n. The latter combined with (*) gives the (C, α) or the $(C, \alpha+)$ summability of s_n. Some inexactitudes were noted in §7: The assumption that $\sum a_n$ is Borel summable should be added in (7.3) and (7.4); it is wrongly asserted that $(C, \alpha+)$ summability of s_n implies that σ_n^α is bounded. *G. G. Lorentz* (Detroit, Mich.).

Yurtsever, Berki. Über die C-Summierbarkeit der unendlichen Reihen. Communications Fac. Sci. Univ. Ankara. Sér. A. 5, 1–11 (1953). (Turkish summary)

Let A be a linear method for evaluation of sequences s_n, a series $\sum a_n$ being evaluable A if its sequence s_n of partial sums is evaluable A. The Abel partial summation formula

$$\sum_{k=0}^{n}a_kb_k=s_nb_{n+1}+\sum_{k=0}^{n}s_k(b_k-b_{k+1})$$

then implies immediately that the series $\sum a_kb_k$ is evaluable A whenever the sequence s_nb_{n+1} and the series $\sum s_k(b_k-b_{k+1})$ are both evaluable A. This result is proved for the special case in which A is the Cesàro method C_1. Suppose now that A is both linear and regular, $\sum a_n$ is evaluable A, $s_n=O(1)$, and $\sum|b_k-b_{k+1}|<\infty$. Then

$$\sum_{k=0}^{n}a_kb_k=bs_n+\epsilon_n+x_n,$$

where $b=\lim b_n$, $\epsilon_n=s_n(b_{n+1}-b)=o(1)$ and

$$x_n=\sum_{k=0}^{n}s_k(b_k-b_{k+1})=o(1)+\sum_{k=0}^{\infty}s_k(b_k-b_{k+1}),$$

and the hypotheses imply that $\sum a_kb_k$ is evaluable A. A special case of this result is obtained for the special case in which A is C_1. *R. P. Agnew* (Ithaca, N. Y.).

Borwein, D. Integration by parts of Cesàro summable integrals. J. London Math. Soc. 29, 276–292 (1954).

Let $\lambda\geqq 1$. Conditions on a function $\phi(t)$ are given and, with suitable interpretations of the symbols, are shown to be sufficient for validity of the formula

$$C_\lambda\left\{\int_1^\infty f(t)\phi(t)dt\right\}$$

$$=[\{f_1(t)-s\}\phi(t)]_1^\infty-C_{\lambda-1}\left\{\int_1^\infty[f_1(t)-s]\phi'(t)dt\right\}$$

whenever $f(t)$ is a function which is integrable over each finite interval and is such that the left member exists. The function $f_1(t)$ is defined by $f_1(t)=\int_1^t f(u)du$, and the statement $C_r\{\int_0^\infty F(t)dt\}=A$ means that the integral $\int_0^\infty F(t)dt$ is evaluable to A by the Cesàro method C_r. The conditions on $\phi(t)$ are milder than those heretofore used. Some problems involving necessity of the conditions are solved. *R. P. Agnew* (Ithaca, N. Y.).

Belgrano Bremard, J. C. Bounds for the remainders in the Euler-Maclaurin formulas. Revista Mat. Hisp.-Amer. (4) 13, 320–327 (1953). (Spanish)

Bounds for the remainders of index k are obtained under certain assumptions concerning the sign and monotonicity of the derivatives $f^{(2k+j)}(x)$ $(j=1, 2, 3)$. *F. Bagemihl.*

Agmon, S. On the singularities of a class of Dirichlet series. Bull. Res. Council Israel 3, 385–389 (1954).

Die Dirichlet-Reihe

$$f(s)=\sum_0^\infty a_ne^{-\lambda_ns}\text{ mit }0\leqq\lambda_n\uparrow\infty\text{ und lim inf }(\lambda_{n+1}-\lambda_n)=h>0$$

sei in $\Re(s)>0$ konvergent; die maximale Dichte von $\{\lambda_n\}$ sei $D_\lambda=\lim_{\xi\to 1-}\limsup_{r\to\infty}[N(r)-N(\xi r)]/(r-\xi r)$, wobei $N(r)$ die Anzahl der $\lambda_n\leqq r$ ist. Ferner seien in einem Intervall I auf $\Re(s)=0$ der Länge $>2\pi(D_\lambda+h^{-1})$ nur einfache Pole als Singularitäten; diese Pole liegen in $s=i\alpha_q$ $(q=1, 2, \cdots, k)$. Schliesslich bezeichne S_f die Menge aller singulären Stellen von $f(s)$ auf $\Re(s)=0$. Dann lässt sich $f(s)$ für ein $\delta>0$ eindeutig in $\{\Re(s)>-\delta\}\cap S_f$ fortsetzen; jede isolierte Singularität $i\alpha$ von $f(s)$ auf $\Re(s)=0$ ist überdies auch ein einfacher Pol und es gilt $\alpha=m_1\alpha_1+m_2\alpha_2+\cdots+m_k\alpha_k$ (m_q ganz). Für Reihen der angegebenen Form bestimmt also der Charakter von $f(s)$ in I das Verhalten von $f(s)$ auf der ganzen Kon-

vergenzgeraden weitgehend, wie das ja im Falle $\lambda_n=n$ offensichtlich ist. Beim Beweis wird vor allem die vom Verf. entwickelte 'Methode der Hauptindizes' verwendet [Trans. Amer. Math. Soc. **74**, 444–481 (1953); diese Rev. **14**, 869]. Verf. bemerkt, dass sich die Grösse $2\pi(D_\lambda+h^{-1})$ durch $4\pi D_\lambda$ ersetzen lässt und dass ähnliche Resultate auch im Falle mehrfacher Pole gültig sind. *D. Gaier.*

Snehlata. On the singularities of a class of Dirichlet's series. Proc. Nat. Acad. Sci. India. Sect. A. **20**, 92–102 (1951).

The author proves a theorem on the addition of singularities of familiar type. If $\phi_1(z)$ and $\phi_2(z)$ are entire functions of exponential type, if

$$h_1(s)=\sum_1^\infty \phi_1(\lambda_n)e^{-\lambda_n s} \quad \text{and} \quad h_2(s)=\sum_1^\infty \phi_2(\lambda_n)e^{-\lambda_n s}$$

have their singularities belonging to the sets $\{\alpha_n\}$ and $\{\beta_n\}$ respectively, then the singularities of

$$h(s)=\sum_1^\infty \phi_1(\lambda_n)\phi_2(\lambda_n)e^{-\lambda_n s}$$

belong to the sum set $\{\alpha_n+\beta_m\}$. Further there is a generalization of a classical theorem of H. Cramér [Ark. Mat. Astr. Fys. **13**, no. 22 (1918)]: Suppose that $h_2(s)$ can be continued analytically in a connected domain D_2 containing the half-plane of convergence of the series, let Σ_1 be the convex curve whose function of support is the indicator of $\phi_1(z)$ and let $D_2(\phi_1)$ be the component, containing a half-plane, of the open set obtained by deleting from D_2 the points lying inside or on the curves $s+\Sigma_1$ where s describes the boundary of D_2. Then $h(s)$ is holomorphic in $D_2(\phi_1)$. *E. Hille.*

Panday, Nirmala. A class of Dirichlet's series possessing essential characteristics of a Taylor's series. Proc. Nat. Acad. Sci. India. Sect. A. **21**, 17–23 (1952).

The author discusses Dirichlet's series of the form $\sum\lambda'(n)f[\lambda(n)]e^{-s\lambda(n)}$, previously considered by P. L. Srivastava in a paper of the same title [Bull. Acad. Sci. Allahabad, **3**, 11–16 (1933)]. Here $f(z)$ is an entire function of order one with indicator $A\cos\theta+B\sin\theta$, $\lambda(z)$ is analytic in the plane slit along the real axis from $-\infty$ to β, uniformly $o(|z|)$ as $z\to\infty$ with $\lambda'(z)=o(1)$; further, $\lambda(x)$ is a positive monotone unbounded L-function for large x. Assuming in addition that (1) $f[\lambda(z)]$ is entire and

$$\text{(2)} \qquad z\lambda'(z)f[\lambda(z)]e^{-k\lambda(z)}\to 0,$$

for some $k>0$, uniformly in z as $z\to\infty$ in the sector, the author shows that the function defined by the series has $z=A-Bi$ as its only singularity. [Rev. The paper is marred by many disturbing misprints. It is hard to satisfy conditions (1) and (2). The choice $\lambda(z)=\log z$, $f(z)=e^{mz}$, m integer, is admissible but trivial. The author proposes for $\lambda(z)$ one of the functions z^α $(0<\alpha<1)$, $z/(\log z)$ and $z/(\log\log z)$. The latter two functions are inadmissible and so is z^α for $\frac{1}{2}\leqq\alpha<1$ since the conditions require $f(z)\equiv 0$. If α is rational, $\alpha=p/q<\frac{1}{2}$, $f(z)=g(z^q)$, where $g(w)$ is any entire function of order $1/q$ and minimal type, but if α is irrational $<\frac{1}{2}$ only $f(z)\equiv$const. will do. In both cases $A=B=0$.] *E. Hille* (New Haven, Conn.).

Gonçalves, J. Vicente. Sur la décomposition de cot x. Univ. Lisboa. Revista Fac. Ci. A. Ci. Mat. (2) **3**, 200–202 (1954).

Farinha, João. Periodic ascending continued fractions. Revista Fac. Ci. Univ. Coimbra **22**, 110–113 (1953). (Portuguese)

The author shows that if $|a_i|>1$, $i=1, 2, \cdots, k$, then the k-periodic ascending continued fraction

$$\overset{\infty}{\underset{1}{\Phi}} a_i\backslash b_i=\cfrac{b_1+\cfrac{b_2+\cdots}{a_2}}{a_1}$$

converges and its value is $R_kB_k[B_k-1]^{-1}$, where $R_k=\Phi_1^k a_i\backslash b_i$ and $B_k=a_1a_2\cdots a_k$. *H. S. Wall* (Austin, Tex.).

Perron, Oskar. Über die Preece'schen Kettenbrüche. S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. **1953**, 21–56 (1954).

The author gives a method for evaluating certain continued fractions and illustrates the method by deriving formulas of Stieltjes, Rogers, Ramanujan, and Preece. A given convergent continued fraction containing a parameter x and having the value $F(x)$ is transformed by the method of Bauer and Muir to yield a new continued fraction. The transformation is so chosen (if possible, and it is possible in the illustrative examples considered) that (1) the new continued fraction converges to $F(x)$, and (2) the form of the new continued fraction requires that $F(x)$ satisfy a functional relation

$$F(x)=\frac{P(x)F(x+\gamma)+Q(x)}{R(x)F(x+\gamma)+S(x)}.$$

Iteration of the functional relation leads to an evaluation of the original continued fraction. *W. T. Scott.*

Fourier Series and Generalizations, Integral Transforms

Rogosinski, W. W. Extremum problems for polynomials and trigonometrical polynomials. J. London Math. Soc. **29**, 259–275 (1954).

Consider the class of real trigonometric polynomials

$$t(\theta)=a_0+\sum_{k=1}^n(a_k\cos k\theta+b_k\sin k\theta), \quad |t(\theta)|\leqq 1.$$

Let $I(t)=\sum_{k=0}^n(\lambda_ka_k+\mu_kb_k)$ and let $\max|I(t)|$ be required. Many familiar problems are of this form and have been attacked by special devices. The author gives a general theory which enables him to find all $I(t)$ which have a given extremal $t(\theta)$ (though not necessarily to find $\max|I(t)|$ for a given I). By selecting various extremal $t(\theta)$ he solves a number of particular extremal problems. The main theorem is as follows. It is shown (by using functional analysis) that $I(t)=\int_0^{2\pi}t(\theta)d\mu(\theta)$ with $\max|I(t)|=\int_0^{2\pi}|d\mu(\theta)|$. With $\mu(\theta)$ suitably normalized, if $T(\theta)\equiv 1$ is an extremal polynomial for I, the corresponding extremal $\mu(\theta)$ (maximizing I) increases; if $T(\theta)$ is extremal and not ± 1, the extremal $\mu(\theta)$ is uniquely determined as a nonconstant step function with at most n points α_j of increase and n points β_j of decrease, in $(0, 2\pi]$, and $T(\alpha_j)=1$, $T(\beta_j)=-1$. In particular, the λ_k, μ_k for which $\cos s\theta$ is extremal are characterized. An equivalent problem is to maximize $J(p)=\sum_{k=0}^n\gamma_kc_k$, where $p(z)=\sum_{k=0}^n c_kz^k$, c_0 real, $|\Re p(z)|\leqq 1$ in $|z|\leqq 1$. For example,

$|c_s| \leq 1$ if $3s > n$ [cf. van der Corput and Visser, Nederl. Akad. Wetensch., Proc. 49, 383–392 (1946); these Rev. 8, 148]. The theory does not determine $C_k(n) = \max|c_k|$ (for $p(z)$ of degree at most n), for $k < n/3$; but $C_k(n) = C_1([n/k])$, and the author gives a special discussion to show that $C_1(n) \uparrow 4/\pi$ as $n \uparrow \infty$, and that $C_1(3) = C_1(4) = 2/\sqrt{3}$.

R. P. Boas, Jr. (Evanston, Ill.).

Rogosinski, W. W. Linear extremum problems for real polynomials and trigonometrical polynomials. Arch. Math. 5, 182–190 (1954); corrigenda 6, 87 (1955).

If $p(x) = a_0 + \cdots + a^n x^n$ is a real polynomial of degree at most n the problem considered is to determine, for given real λ_k, $\max|\lambda_0 a_0 + \cdots + \lambda_n a_n|$ over the class of $p(x)$ for which $\|p\|^\alpha = \int_E W(x)|p(x)|^\alpha dx \leq 1$, with a given set E, positive W, and $\alpha \geq 1$ (including $\alpha = \infty$ with the usual convention); the analogous problem is considered for trigonometric polynomials. The author gives a general theory of the existence and uniqueness of extremal solutions, using tools from functional analysis; applications are reserved for later study. For $1 < \alpha < \infty$ there is a unique extremal polynomial, and a given polynomial P is extremal if and only if $\lambda_k = A\int_E x^k W \operatorname{sgn} P \cdot |P|^{\alpha-1} dx$. For $\alpha = 1$ and ∞ the results are more complicated and cannot be summarized briefly.

R. P. Boas, Jr. (Evanston, Ill.).

Stečkin, S. B. On the theorem of Kolmogorov-Seliverstov. Izvestiya Akad. Nauk SSSR. Ser. Mat. 17, 499–512 (1953). (Russian)

Suppose $f \in L_2(0, 2\pi)$ with norm $\|f\|$, and S_n are the partial sums of the Fourier series $S[f]$ of f. The author establishes several conditions for the convergence almost everywhere of $S[f]$, each of which is equivalent to the criterion of Kolmogorov-Seliverstov and to that of Plessner. They are of the form

$$(*) \qquad \sum_{n=1}^{\infty} \frac{1}{n} M_n^2(f) < +\infty,$$

where M_n is equal to $\|f - S_n\|$, $\sup_{0 \leq t \leq 1/n} \|f(x+t) - f(x-t)\|$, or $\sup_{0 \leq t \leq 1/n} \|f(x+t) - 2f(x) + f(x-t)\|$, respectively; and for each a local analog is given. For example, the convergence of $S[f]$ almost everywhere in $[a, b]$ is guaranteed if (*) holds with $M_n^2 = \int_a^b |f(x) - S_n(x)|^2 dx$.

G. Klein.

Ul'yanov, P. L. Generalization of a theorem of Marcinkiewicz. Izvestiya Akad. Nauk SSSR. Ser. Mat. 17, 513–524 (1953). (Russian)

The criterion of Marcinkiewicz for the convergence almost everywhere of the Fourier series of a function $f \in L_p(0, 2\pi)$, $1 \leq p \leq 2$, namely that

$$\int_0^{2\pi} \int_0^{2\pi} t^{-1} |f(x+t) - f(x-t)|^p dt dx < +\infty,$$

is extended so as to reflect the local character of that convergence. In effect, this extension is obtained from the original by setting $f(x) = 0$ outside an arbitrary subinterval $[a, b]$ and modifying it slightly near the end-points, and insures convergence almost everywhere in $[a, b]$. For $p = 2$, following Plessner, the so obtained criterion is shown equivalent to the condition $\sum_{k=2}^{\infty} (a_k^2 + b_k^2) \log k < +\infty$ of Kolmogorov-Seliverstov as applied to the product of f and the characteristic function of $[a, b]$.

G. Klein.

Livingston, Arthur E. The Lebesgue constants for Euler (E, p) summation of Fourier series. Duke Math. J. 21, 309–313 (1954).

Let $0 < r < 1$. The Lebesgue constant of the Euler $(E, (1-r)/r)$ method of summation is

$$L_n = \frac{2}{\pi} \int_0^{\pi/2} x^{-1} (1 - 4r(1-r) \sin^2 x)^{n/2} |\sin 2\pi x r| dx + o(1) \quad \text{as } n \to \infty.$$

The author proves that

$$L_n = \frac{2}{\pi} \log \frac{2nr}{1-r} + A + o(1) \quad \text{as} \quad n \to \infty,$$

where

$$A = -\frac{2}{\pi^2} C + \frac{2}{\pi} \int_0^1 \frac{\sin u}{u} du - \frac{2}{\pi} \int_1^\infty \left(\frac{2}{\pi} - |\sin u| \right) \frac{du}{u},$$

C being the Euler constant. The case $r = \frac{1}{2}$ was proved by Lee Lorch [same J. 19, 45–50 (1952); these Rev. 13, 645].

S. Izumi (Tokyo).

Heywood, P. On the integrability of functions defined by trigonometric series. Quart. J. Math., Oxford Ser. (2) 5, 71–76 (1954).

Let $f(x)$, $g(x)$ denote the sums of the series

$$\tfrac{1}{2}\lambda_0 + \sum_{n=1}^{\infty} \lambda_n \cos nx, \quad \sum_{n=1}^{\infty} \lambda_n \sin nx.$$

The author proves that (1) if $\lambda_n \downarrow 0$, $x^{-\gamma} g(x)$ is $L(0, \pi)$, $1 < \gamma < 2$, if and only if $\sum n^{\gamma-1} \lambda_n$ converges; (2) if λ_n is ultimately nonnegative and if $\frac{1}{2}\lambda_0 + \sum_1^\infty \lambda_n$ converges to 0, then $x^{-\gamma} f(x)$ is $L(0, \pi)$, $1 < \gamma < 3$, if and only if $\sum n^{\gamma-1} \lambda_n$ converges. For $\gamma = 0$, (1) was proved by W. H. Young; for $0 < \gamma \leq 1$, by the reviewer [same J., same ser. (2) 3, 217–221 (1952); these Rev. 14, 867]; for $0 < \gamma < 1$, a result similar to (2) was proved by the reviewer [loc. cit.]; for $\gamma > 2$, (1) fails, while (2) fails if $\gamma = 1$ or $\gamma \geq 3$. [The reviewer's result about sine series, and apparently also that of the present paper, are effectively special cases of results of S. M. Edmonds, Proc. Cambridge Philos. Soc. 46, 249–267 (1950); these Rev. 11, 592.]

R. P. Boas, Jr. (Evanston, Ill.).

Takahashi, Takehito. A problem on Fourier series. Bull. Earthquake Res. Inst. Tokyo 32, 147–153 (1954). (Japanese summary)

Given $f(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$, distributions are obtained with Fourier series

$$\sum_{n=1}^{\infty} \varphi(n)(a_n \cos nx + b_n \sin nx)$$

for $\varphi(n) = n^m$, $m \geq -1$ or r^n, $|r| \leq 1$.

P. Civin.

Titchmarsh, E. C. Some properties of eigenfunction expansions. Quart. J. Math., Oxford Ser. (2) 5, 59–70 (1954).

In a previous paper [same J. (2) 2, 258–268 (1951); these Rev. 13, 654] the author proved that the Fourier expansion

$$f(x) \sim \sum_{n=0}^{\infty} c_n \psi_n(x), \quad c_n = \int_0^\infty f(t) \psi_n(t) dt,$$

associated with a differential problem

$$\psi'' + [\lambda - q(x)]\psi = 0, \quad x \geq 0, \quad \psi(0) \cos \alpha + \psi'(0) \sin \alpha = 0,$$

having a discrete spectrum, is summable $(R, \lambda, 1)$ to $f(x)$

wherever

$$\int_0^\eta |f(x+t)-f(t)|\,dt=o(\eta)$$

provided $f(x)\in L^2(0,\infty)$. Here the series is said to be summable (R,λ,p) to $f(x)$ if

$$\lim_{\Lambda\to\infty}\sum_{\lambda_n<\Lambda}\left(1-\frac{\lambda_n}{\Lambda}\right)^p c_n\psi_n(x)=f(x).$$

In the present paper the restriction $f(x)\in L^2(0,\infty)$ is replaced by $f(x)\in L(0,T)$ for every T and $\int_0^T|f(t)|dt=o(T^a)$ for some a. The author proves summability (R,λ,p) to $f(x)$ in the Lebesgue set for $p\geq\max[1,2(a-1)/(k+2)]$ provided $q(x)$ satisfies: (1) $q(s)$ is holomorphic in a sector containing the positive real axis, (2) in the sector $\Im[q(x+iy)]$ is an increasing function of y, (3) $q(s)\sim As^k$, $k>0$, with corresponding slightly weaker conditions on the first three derivatives, and (4) $q(x)$ is non-decreasing for $x>0$. The proof is based on a refinement of the contour-integration argument used in the previous paper. If $\psi(x,\lambda)$ is the solution of the differential equation in $L^2(0,\infty)$ and $x>A|\lambda|^{1/k}$, it is shown that $\psi(x,\lambda)=O[\varphi(\lambda)\exp(-Ax^{\frac{1}{2}k+1})]$, where $\varphi(\lambda)=|\lambda|^{-1/2}$ or $|\lambda|^{-1/k}\nu^{-1}$, $\lambda=\mu+i\nu$, and the second value holds when $0<\nu<|\lambda|^{1/2-1/k}$. It is the investigation of this "parabolic" neighborhood of the positive real axis in the λ-plane which forms the crux of the problem and requires the analyticity restrictions on $q(s)$. A method due to R. E. Langer [Trans. Amer. Math. Soc. 33, 23–64 (1931)] provides the analytic machinery for this part of the work. Under the same assumptions on $q(x)$ and $f(x)$ it is shown that convergence of $\sum|c_n|^2$ implies $f(x)\in L^2(0,\infty)$. There is also an analogue of the Young-Hausdorff theorem: If $f(x)\in L^{1/\alpha}(0,\infty)$ where $\frac12\leq\alpha<1$, then

$$\left\{\sum_{n=0}^\infty\lambda_n^{h(2-1/\beta)}|c_n|^{1/\beta}\right\}^\beta<M\left\{\int_0^\infty|f(t)|^{1/\alpha}dt\right\}^\alpha,$$

$$h=\frac{1}{12}-\frac{1}{3k},\quad \beta=1-\alpha.$$

Conversely, if $\int_0^T|f(t)|dt=o(T^a)$ and $\sum\lambda_n^{h(2-1/\alpha)}|c_n|^{1/\alpha}<\infty$, then

$$\left\{\int_0^\infty|f(t)|^{1/\beta}dt\right\}^\beta<M\left\{\sum_{n=0}^\infty\lambda_n^{h(2-1/\alpha)}|c_n|^{1/\alpha}\right\}^\alpha.$$

These results are based on the estimate $|\psi_n(x)|<A\lambda_n^h$, which is the best of its kind.

E. Hille.

Kac, M. Toeplitz matrices, translation kernels and a related problem in probability theory. Duke Math. J. 21, 501–509 (1954).

Theorem: Let F be the Fourier transform of an even function ρ such that $\rho(x)\geq0$, $\int_{-\infty}^\infty\rho(x)dx=1$, and suppose that $|x|\rho(x)$ and F belong to $L^1(-\infty,\infty)$. Let $D_a(\lambda)$ denote the Fredholm determinant of the integral equation

$$\int_{-a}^a\rho(x-y)\rho(y)dy=\lambda\rho(x).$$

Then, for sufficiently small real λ,

$$\lim_{a\to\infty}D_a(\lambda)\exp\left\{-\frac{a}{\pi}\int_0^\infty\log[1-\lambda F(\eta)]d\eta\right\}$$

$$=\exp\left\{\int_0^\infty x\left|\frac{1}{2\pi}\int_{-\infty}^\infty\log[1-\lambda F(\eta)]e^{ix\eta}d\eta\right|^2dx\right\}.$$

This theorem is an analogue of a result which Szegö recently proved for Fourier series [Comm. Sém. Math. Univ. Lund [Medd. Lunds Univ. Mat. Sem.] Tome Supplémentaire, 228–238 (1952); these Rev. 14, 553]. An interesting account of the way in which the theorem was discovered is included. Szegö's result is reinterpreted in probabilistic terms, and it is shown that both Szegö's result and the above analogue are consequences of the following combinatorial identity: Let $s=(s_1,\cdots,s_n)$ be a permutation of $(1,\cdots,n)$, let $a_1,\cdots,a_n$ be real numbers, and let $N(s)$ be the number of non-negative terms in the sequence

$$a_{s_1},\ a_{s_1}+a_{s_2},\ \cdots,\ a_{s_1}+\cdots+a_{s_n},$$

whose largest term is $M(s)$. Then

$$\sum_s\max(0,M(s))=\sum_s N(s)a_{s_1},$$

the summations being extended over all permutations of $(1,\cdots,n)$.

The reviewer noticed one minor oversight: Szegö's result is proved by him under the assumption that f' satisfies a Lipschitz condition of order α $(0<\alpha\leq1)$, and is proved in the present paper under the assumption that f' has an absolutely convergent Fourier series: it is claimed that the second condition is weaker than the first; but this is so only if $\alpha>\frac12$.

W. Rudin (Rochester, N. Y.).

Rajagopal, C. T., and Jakimovski (Amir), Amnon. Applications of a theorem of O. Szász for the product of Cesàro and Laplace transforms. Proc. Amer. Math. Soc. 5, 370–384 (1954).

O. Szász has used the device of integrating an absolutely convergent Laplace integral under the integral sign with respect to the parameter, and the first author has used the same idea for integration of fractional order [Szász, Trans. Amer. Math. Soc. 39, 117–130 (1936); Rajagopal, Amer. J. Math. 69, 371–378, 851–852 (1947); these Rev. 9, 26, 278; see also Szász, Proc. Amer. Math. Soc. 3, 257–263 (1952); these Rev. 13, 835]. Here the authors exploit the idea in the form: (1) if

$$\frac{s^{\alpha+1}}{\Gamma(\alpha+1)}\int_0^\infty e^{-su}s(u)u^\alpha du\to C$$

as $s\to+0$, where $\alpha>0$, then the same holds with $\alpha=0$.

They prove some Tauberian theorems of the Hardy-Littlewood type for Laplace integrals which supplement known results. In particular, (2) if

$$\int_0^\infty e^{-su}s(u)du\sim Cs^{-\alpha-1}\quad\text{as}\quad s\to+0,$$

and

$$(*)\qquad s(x)-\frac{\alpha+1}{x}\int_0^x s(u)du\geq-Kx^\alpha,$$

then

$$\int_0^x s(u)du\sim Cx^{\alpha+1}/\Gamma(\alpha+2)\quad\text{as}\quad x\to+\infty.$$

[the case $\alpha=0$ was proved by Szász, loc. cit., 1936]. They observe that the Hardy-Littlewood "positive" theorem (for integrals), in which the Tauberian condition $s(x)\geq-Kx^\alpha$ takes the place of (*), may be deduced from (2).

In view of the fact that the "positive" theorem is usually regarded as the fundamental one in this field it is perhaps relevant to remark that (a) in proving (2) the authors use the case $\alpha=0$ of the "positive" theorem [this case was

formulated for integrals by G. Doetsch, Math. Ann. **82**, 68–82 (1920)], (b) the general form of the "positive" theorem, with $\alpha>0$, may be deduced directly from the case $\alpha=0$ by means of (1). *L. S. Bosanquet* (London).

Pistoia, A. Alcuni teoremi tauberiani per la trasformata doppia di Laplace. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **16**(85), 170–190 (1952).

The author obtains several inversion theorems concerning the double Laplace transform:

$$f(p,q)=\int_0^\infty\int_0^\infty e^{-px-qy}F(x,y)dxdy.$$

Under hypotheses too complicated to state here, he shows that

$$F(x,y)=\lim_{(\sigma,\rho)\to 0}\frac{1}{4\pi^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}e^{i(\alpha x+\beta y)}\times f(i\alpha,i\beta)(1+\rho^2\alpha^2)^{-1}(1+\sigma^2\beta^2)^{-1}d\alpha d\beta,$$

where the limit is taken with $\theta^{-1}\leqq\rho/\sigma\leqq\theta$ ($\theta\geqq 1$). If, in addition, $f(i\alpha,i\beta)$ is summable in the plane, then the following inversion formula holds:

$$(*)\qquad F(x,y)=\frac{1}{4\pi^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}e^{i(\alpha x+\beta y)}f(i\alpha,i\beta)d\alpha d\beta.$$

This formula is also valid under more general hypotheses than summability if the integral is taken as $\lim_{(\lambda,\mu)\to\infty}\int_{-\lambda}^{\lambda}\int_{-\mu}^{\mu}$.

The author uses these theorems to find conditions on $f(i\alpha,i\beta)$ that will insure: $\lim_{(x,y)\to\infty}F(x,y)=0$. These are conditions involving summability of f, or the uniform convergence of (*), or that f have bounded second variation, according to Hardy, in a certain region; there are several theorems of this character. Finally these results are readily extended to get conditions that will make: $\lim_{(x,y)\to\infty}F(x,y)=A$. It should be pointed out that the inversion formulas given in Theorems II and III are valid without requiring all of the hypotheses of Theorem I to be true. *D. L. Bernstein* (Rochester, N. Y.).

Doetsch, Gustav. Über die Singularitäten der Mellin-Transformierten. Math. Ann. **128**, 171–176 (1954).

Let

$$\mathfrak{M}\{\Phi\}=\int_0^\infty z^{s-1}\Phi(z)dz=\varphi(s),\quad \Phi(z)=\frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty}z^{-s}\varphi(s)ds$$

be the Mellin transform of $\Phi(z)$ and its inverse where $\Phi(z)$ and $\varphi(s)$ belong to adjoint classes $\mathfrak{B}$ and $\mathfrak{b}$ resp. Here $\Phi(z)\in\mathfrak{B}$ if $\Phi(z)$ is analytic in a sector $\vartheta_1\leqq\vartheta\leqq\vartheta_2$, except possibly at $z=0$, and satisfies estimates $|\Phi(z)|<C\rho^{-x_1}$, $\rho<1$, and $|\Phi(z)|<C\rho^{-x_2}$, $\rho>1$, $z=\rho e^{i\vartheta}$, $x_1<x_2$. And $\varphi(s)\in\mathfrak{b}$ if $\varphi(s)$ is analytic in a strip $x_1\leqq x\leqq x_2$, $s=x+iy$, and satisfies $|\varphi(s)|<Ce^{-\vartheta_1 y}$, $y<0$, $|\varphi(s)|<Ce^{-\vartheta_2 y}$, $y\geqq 0$, $\vartheta_1<\vartheta_2$. If $\Phi(z)\in\mathfrak{B}$, then $\varphi(s)$ exists in $\mathfrak{b}$ and vice versa. The functions of $\mathfrak{B}$ form an algebra under multiplication, those of $\mathfrak{b}$ under complex convolution

$$\varphi(s)=\frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty}\varphi_1(\sigma)\varphi_2(s-\sigma)d\sigma\equiv(\varphi_1*\varphi_2)(s).$$

The author shows that if $\varphi_1(s)$, $\varphi_2(s)\in\mathfrak{b}$ and are meromorphic in the left half-planes contiguous to their strips of holomorphism, then so is $\varphi_1*\varphi_2=\varphi(s)$, the poles of $\varphi(s)$ belong to the "sum-set" of the sets of poles of $\varphi_1(s)$ and of $\varphi_2(s)$, and the principal parts of $\varphi(s)$ are obtained from the principal parts of $\varphi_1(s)$ and $\varphi_2(s)$ by a composition. Thus the pole $s=-\mu$ of $\varphi_1(s)$ with principal part $\sum c_m(s+\mu)^{-m}$ combines with the pole $s=-\nu$ of $\varphi_2(s)$ with principal part $\sum d_n(s+\nu)^{-n}$ to give a pole $s=-\mu-\nu$ of $\varphi(s)$ with principal part

$$\sum\sum\binom{m+n-2}{m-1}c_m d_n(s+\mu+\nu)^{-m-n+1}.$$

If several poles give the same sum $-\mu-\nu$, the principal part of $\varphi(s)$ is the sum of the corresponding principal parts. If the principal part should turn out to be zero, the corresponding point $-\mu-\nu$ is regular. The proof is based on interesting theorems relating the poles of $\mathfrak{M}\{\Phi(z)\}$ with the asymptotic representation of $\Phi(z)$ at $z=0$ and conversely (proved in special cases by Hj. Mellin). Thus if $\Phi(z)$ is holomorphic in $|\vartheta|\leqq\vartheta_0$ and

$$\Phi(z)\sim\sum_m\Big[\sum_k b_m{}^k(-\log z)^k\Big]z^{\lambda_m},\quad \Re(\lambda_m)<\Re(\lambda_{m+1}),$$

as $z\to 0$ and $\Phi(z)=O(z^\tau)$, $\tau<\Re(\lambda_0)$, for $z\to\infty$, then $\varphi(s)$ is meromorphic in $\Re(s)<-\tau$, its poles are at $s=-\lambda_m$, where the principal part is $\sum_m b_m{}^k k!(s+\lambda_m)^{-k-1}$ and $|\varphi(s)|<Ce^{-\vartheta_0|t|}$ in any bounded vertical strip omitting δ-neighborhoods of the poles. The converse also holds. *E. Hille.*

Churchill, R. V. The operational calculus of Legendre transforms. J. Math. Physics **33**, 165–178 (1954).

The sequence of numbers $f(n)$ defined by the equation

$$T\{F(x)\}=f(n)=\int_{-1}^{+1}F(x)P_n(x)dx,$$

where $P_n(x)$ denotes the Legendre polynomial of degree n, is the Legendre transform $T\{F(x)\}$ of the function $F(x)$. For functions $F(x)$ satisfying certain conditions on the interval $-1\leqq x\leqq 1$, there exists the inversion formula

$$F(x)=\sum_{n=0}^{\infty}(n+\tfrac12)f(n)P_n(x)=T^{-1}\{f(n)\}.$$

When we denote

$$R(F(x))=\frac{d}{dx}\Big[(1-x^2)\frac{d}{dx}F(x)\Big],$$

the following result is obtained under very general conditions:

$$T\{R(F(x))\}=-n(n+1)f(n),\quad n=0,1,2,\cdots.$$

This formula represents the basic operational property of the Legendre transformation T. The author gives several properties of this transformation, especially a convolution theorem and a short table of Legendre transforms. The operational calculus is applied to some boundary-value problems in partial differential equations. *W. Saxer.*

Churchill, R. V., and Dolph, C. L. Inverse transforms of products of Legendre transforms. Proc. Amer. Math. Soc. **5**, 93–100 (1954).

Per la trasformazione di Legendre, la quale ammette applicazioni simili a quella di Laplace, gli autori stabiliscono una regola di composizione. Cioè, definite le trasformate

$$T\{F(x)\}=\int_{-1}^{1}F(x)P_n(x)dx=f(n),$$

$$T\{G(x)\}=\int_{-1}^{1}G(x)P_n(x)dx=g(n),$$

essendo $P_n(x)$ il polinomio di Legendre di ordine n ($n=0, 1, 2, \cdots$), essi esprimono la funzione $H(x)$, indicata con $F(x) * G(x)$, per la quale si ha $T\{H(x)\}=f(n)g(n)$. L'espressione di $H(x)$ viene data senza far ricorso alla nota formula d'inversione per le trasformate di Legendre. Partendo dalla nota relazione

$$P_n(\cos \lambda)P_n(\cos \mu)=\frac{1}{\pi}\int_0^\pi P_n(\cos \nu)d\alpha,$$

ove $\cos \nu=\cos \lambda \cos \mu+\sin \lambda \sin \mu \cos \alpha$, vengono dedotte cinque diverse formule per $H(x)$. La più semplice è

$$H(x)=\frac{1}{\pi}\iint_{E(x)} F(y)G(z)(1-x^2-y^2-z^2+2xyz)^{-\frac{1}{2}}dydz,$$

ove $E(x)$ è l'interno dell'ellisse $y^2+z^2-2xyz=1-x^2$. Le funzioni $F(x)$ e $G(x)$ si suppongono limitate e integrabili nel senso di Riemann. *M. J. De Schwarz* (Roma).

Lafleur, Charles. Sur l'emploi de la fonction de Dirac et de ses dérivées et le théorème des résidus de Cauchy. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 40, 701–704 (1954).

La convolution d'une fonction avec la fonction de Dirac ou ses dérivées, cas particuliers du théorème des résidus de Cauchy. *Résumé de l'auteur.*

Special Functions

Orts, J. M.ª. On a theorem of Poincaré. Revista Mat. Hisp.-Amer. (4) 14, 44–49 (1954). (Spanish)

One may deduce $\lim P_{n+1}(z)/P_n(z)$ as $n\to\infty$ from a well-known general theorem of Poincaré on second-order linear difference equations. (The P_n are Legendre polynomials and z is complex.) The author shows that this limit may also be obtained from the trigonometric expansion of Legendre polynomials. He also proves that $P_n(x)$ and $P_{n+1}(x)/P_n(x)$ are increasing functions of n when $x>1$. *A. Erdélyi.*

Erdélyi, A., Kennedy, M., and McGregor, J. L. Parabolic cylinder functions of large order. J. Rational Mech. Anal. 3, 459–485 (1954).

A comprehensive discussion of the solutions of the differential equation

$$(*)\qquad \frac{d^2y}{dx^2}-4\nu^2(x^2-1)y=0$$

is given when $|\nu|$ is large, x and ν complex. The authors' interest in (*) stems partly from physical applications, e.g. wave motion in a parabolic cylinder, and partly from the fact that it is perhaps the simplest example of a second-order equation having two turning points. The results are to be used in a general discussion of such equations.

The method used is that of Langer as modified by Cherry [Trans. Amer. Math. Soc. 68, 224–257 (1950); these Rev. 11, 596]. The equation (*) is compared with the Airy equation whose solutions are found to represent asymptotically those of (*). Langer's theory enables one to discuss solutions of (*) only in regions containing a single one of the turning points $x=\pm 1$. Here, asymptotic representations for each of the four solutions $D_{\nu-\frac{1}{2}}(\pm 2x\sqrt{\nu})$, $D_{-\nu-\frac{1}{2}}(\pm 2ix\sqrt{\nu})$ are given in each of several regions of the x-plane which together cover the entire plane. ($D_n(z)$ is the parabolic cylinder function of order n.) The representations are uniform in x as $\nu\to\infty$ on a ray $\arg(\nu)=\text{const}$. The approximations are explicit to terms involving ν^{-1}. More careful attention to the details of the analysis could hardly have been paid, with the result that the paper is genuinely readable. *N. D. Kazarinoff* (Lafayette, Ind.).

Fenyő, István. Über einige unendliche Reihen die mit den Besselschen Funktionen in Bezug sind. Mat. Lapok 4, 277–283 (1953). (Hungarian. Russian and German summaries)

Let x, y be two sides of a triangle and θ the angle between them, let R be the third side, and Ψ the angle opposite y, of the triangle, and let m be an integer. The author expands the integral

$$\frac{1}{2\pi}\int_0^\pi R^\nu \cos(\nu\Psi-m\theta)d\theta$$

in the form

$$\sum_{i=1}^{\infty}\frac{J_{\nu+m}(\lambda_{\nu i}x)J_m(\lambda_{\nu i}y)}{\lambda_{\nu i}J_{\nu+1}(\lambda_{\nu i})},$$

where the $\lambda_{\nu i}$ are the positive zeros of $J_\nu(z)$ and $0<x, y<\frac{1}{2}$. Several series of Bessel functions are summed by means of this result. *A. Erdélyi* (Pasadena, Calif.).

Harmonic Functions, Potential Theory

Sheffer, I. M. A class of functions related to harmonic functions. Duke Math. J. 21, 479–489 (1954).

Le disque $K=K(\xi, \eta; a)$ est donné une fois pour toutes; soit ($\mathcal{P}$) la classe des fonctions $u(x, y)$ analytiques dans K et dont toutes les dérivées partielles $u_{ij}(x, y)=\partial^{i+j}u(x, y)/\partial x^i\partial y^j$ ont un développement en série entière uniformément convergeant sur le disque fermé $\bar{K}$. Problème: déterminer les $u \in (\mathcal{P})$ telles que toutes les u_{ij} satisfont au théorème de la moyenne de Gauss pour la frontière K^*, i.e.:

$$u_{ij}(\xi, \eta)=\frac{1}{2\pi}\int_0^{2\pi} u_{ij}(\xi+a\cos t, \eta+a\sin t)dt \quad (i, j=0, 1, \cdots).$$

Les fonctions harmoniques de la classe ($\mathcal{P}$) sont évidemment des solutions du problème, mais il y en a bien d'autres: l'auteur montre que celles-ci sont essentiellement les fonctions vérifiant:

$$\sum_{k=1}^{\infty}\frac{1}{(k!)^2}\left(\frac{a^2}{4}\right)^k \Delta^k u(x, y)=0,$$

où Δ^k est le laplacien itéré k fois; cela permet d'en déterminer explicitement un grand nombre. *J. Deny.*

Myrberg, Lauri. Über die Integration der Differentialgleichung $\Delta u=c(P)u$ auf offenen Riemannschen Flächen. Math. Scand. 2, 142–152 (1954).

The following theorem is established: Let c denote a C' non-negative density ($\not\equiv 0$) defined on a given non-compact Riemann surface F. Then there exists a C'' positive (non-constant) function u on F satisfying $\Delta u=cu$. The proof depends on the anterior results of Lichtenstein concerning the corresponding problem for the interior of the unit circle [Rend. Circ. Mat. Palermo 33, 201–211 (1912)], and a number of lemmas which permit the author to treat first the problem for a relatively compact region of F with smooth boundary and thereupon to obtain a sequence of

functions defined on regions of F which converges to a solution of the given equation on the whole surface. These lemmas include a maximum principle for functions satisfying the given equation, comparison theorems (with respect to harmonic functions), an analogue of the Harnack principle, and a theorem stating that bounded families of solutions of the given equation are normal. Reference is made to related work of Ozawa [Kōdai Math. Sem. Rep. **1952**, 63–76; these Rev. **14**, 462]. *M. Heins* (Providence, R. I.).

Myrberg, Lauri. Über die Existenz der Greenschen Funktion der Gleichung $\Delta u=c(P)\cdot u$ auf Riemannschen Flächen. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. **170**, 8 pp. (1954).

This paper continues the investigations described in the preceding review. The principal results are the following. (1) Every non-compact Riemann surface admits a Green's function for $\Delta u=cu$. (2) The Green's function enjoys the minimal property of the Green's function for the Laplace equation. (3) Compact surfaces admit positive solutions of $\Delta u=cu$ which are regular save at an assigned point at which they have a logarithmic singularity. In addition, the existence theorem of the preceding paper is given a new proof by use of the device (employed by R. S. Martin in connection with harmonic functions) of considering a sequence of functions $G(P, P_n)/G(P_0, P_n)$ where $G(P, P_n)$ is the Green's function for the surface associated with $\Delta u=cu$ with pole at P_n, P_0 is a fixed point of the surface, and P_n tends to the ideal boundary. There exists a subsequence tending to a non-trivial solution of $\Delta u=cu$. *M. Heins* (Providence, R. I.).

Szegö, G. On the singularities of zonal harmonic expansions. J. Rational Mech. Anal. **3**, 561–564 (1954).

Let (r, θ, φ) denote a spherical coordinate system in 3-space, and suppose $u(r, \theta, \varphi)$ is harmonic in $r<1$ and independent of φ. The author reduces the problem of determining the regular or singular character of a boundary point to the study of an associated power series, by means of the following theorem: If $u(r, \theta)=\sum_0^\infty a_n r^n P_n(\cos\theta)$, where P_n is the nth Legendre polynomial, and $f(\zeta)=\sum_0^\infty a_n\zeta^n$, the point $(1, \theta)$ is a regular point of $u(r, \theta)$ if and only if the point $\zeta=e^{i\theta}$ is a regular point of $f(\zeta)$. *W. Rudin.*

Hong, Imsik. On some boundary value problem in an annulus. Kōdai Math. Sem. Rep. **1954**, 4–6 (1954).

The author considers the harmonic Neumann problem for a circular ring R, in the case in which the boundary values of the normal derivative are continuous along each of a countable number of open subarcs A of the boundary C of R, and $C-A$ is of capacity zero. By a procedure which amounts to considering the corresponding boundary-value problem for the universal covering surface p' of p and mapping p' onto a half-plane, the author reduces the original problem to one related to the half-plane, which is then solved by a method due to L. Myrberg [Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. **103** (1951); these Rev. **13**, 743]. *Z. Nehari* (Pittsburgh, Pa.).

Komatu, Yûsaku, and Hong, Imsik. A mixed boundary value problem for an annulus. Kōdai Math. Sem. Rep. **1954**, 1–3 (1954).

The problem under discussion is the determination of a harmonic function u in $\rho<|z|<1$ from the values of u and $\partial u/\partial n$ on $|z|=1$ and $|z|=\rho$, respectively. The authors observe that u may be written in the form $u=u_1+u_2$ such that $u_1=0$ on $|z|=1$ and $\partial u_2/\partial n=0$ on $|z|=\rho$. Therefore u_1 and u_2 may be continued across $|z|=1$ and $|z|=\rho$, respectively, by inversion. This leads to a Dirichlet problem for u_1 in $\rho<|z|<\rho^{-1}$, and to a Neumann problem for u_2 in $\rho^2<|z|<1$, both of which are solved by means of Villat's formula. *Z. Nehari* (Pittsburgh, Pa.).

Brelot, M., et Choquet, G. Espaces et lignes de Green. Ann. Inst. Fourier Grenoble **3** (1951), 199–263 (1952).

A study is made of certain potential-theoretical problems on connected manifolds $\mathcal{E}$ generalizing Riemann surfaces. The manifolds considered fall into two classes $\mathcal{E}_i$ and $\mathcal{E}_c$, according to the hypotheses on the transfer mapping $H_{P_2}H_{P_1}^{-1}$, where H_{P_1} and H_{P_2} are homeomorphisms of intersecting neighborhoods $\mathcal{V}_{P_1}$ and $\mathcal{V}_{P_2}$ onto neighborhoods of τ-dimensional Euclidean space R^τ $(\tau\geqq 2)$. The manifolds $\mathcal{E}_i$ are those for which $H_{P_2}H_{P_1}^{-1}$ is an isometry of $H_{P_1}[\mathcal{V}_{P_1}\cap\mathcal{V}_{P_2}]$ onto $H_{P_2}[\mathcal{V}_{P_1}\cap\mathcal{V}_{P_2}]$; the manifolds $\mathcal{E}_c$, defined only when $\tau=2$, are those for which $H_{P_2}H_{P_1}^{-1}$ is (directly or inversely) conformal. In each case the space $\mathcal{E}$ (which is connected, locally connected, and locally compact) is shown to be metrizable and hence separable. If the space is not already compact, it can be rendered compact by the adjunction of a point $\mathfrak{A}$.

The Dirichlet problem, Green's function, and the Green's potential are discussed first of all for relatively compact subregions of $\mathcal{E}$. For the most part this discussion consists of a summary of those concepts and results which carry over from the case of bounded regions in n-space. Quite apart from its present interest, this section serves as an excellent reference for the basic ideas relating to the Dirichlet problem (e.g. thin sets, regularity, subharmonic continuation, harmonic measure, resolutivity). However, even in the realm of such classical material the authors make an important contribution by treating the question of resolutivity for arbitrary boundary functions without using the notion of regularity of boundary points. The advantages in this approach become evident when the Dirichlet problem is discussed for regions which are not relatively compact, since the boundary point $\mathfrak{A}$ cannot be classified either as regular or irregular.

For regions not relatively compact the starting points is the following generalization of a theorem of Ohtsuka [Nagoya Math. J. **3**, 91–137 (1951); these Rev. **13**, 642]: in $\mathcal{E}$ the upper envelope of the Green's functions G_P^ω (P fixed and ω an arbitrary relatively compact subregion) is either identically $+\infty$ or harmonic outside of P, the latter case occurring if and only if $\mathcal{E}$ tolerates nonconstant positive superharmonic functions. In this case there exists a Green's function on $\mathcal{E}$, and $\mathcal{E}$ is called a Green's space. The property of being a Green's space is hereditary $(\subset)$, and the usual properties of Green's potentials carry over to Green's potentials defined on such spaces. In treating the Dirichlet problem for Green's spaces, the authors show (without making use of the concept of regularity of boundary points) that a boundary function f is resolutive if and only if it is summable with respect to harmonic measure. Moreover, the solution of the Dirichlet problem is given in the classical way as an integral with respect to harmonic measure: $H_f(M)=\int f d\mu^M$. A number of criteria are given for $\mathcal{E}$ to be a Green's space. For example, $\mathcal{E}$ is a Green's space if and only if it is noncompact and for some (or equivalently, all) compact nonpolar subsets K the point $\mathfrak{A}$ has nonzero harmonic measure relative to $\mathcal{E}-K$.

The Green's functions are studied at length, and the notion of a Green's filter is introduced as follows. For G_P the Green's function on a region $\Omega(\subset\mathcal{E})$ with pole at P, the sets $\{M: G_P(M)<\epsilon\}$ for arbitrary $\epsilon>0$ form a filter base. The corresponding filter, independent of P, is called a Green's filter on Ω. Regular filters, i.e. those finer than the Green's filter, are examined in detail.

A Green's arc is an open arc defined relative to a Green's function G_P, so that on parametric neighborhoods the arc admits a nonzero tangent vector parallel to grad G_P, the latter being assumed nonzero along the arc. A maximal Green's arc is called a Green's line. Locally, the Green's lines behave as follows: through each point of some deleted neighborhood of the pole P there passes a unique Green's line; every Green's line has P as an end-point and is tangent to a corresponding ray issuing from P. The correspondence is, in fact, one-one and establishes a homeomorphism between the family of rays and the points of the level surface $G_P=\lambda$ for λ sufficiently large. Under this correspondence the harmonic measure at P of a Borel set e on the level surface is proportional both to the flux $\int_e(dG_P/dn)d\sigma$ and to the solid angle formed at P by the rays.

The family $\mathcal{L}$ of Green's lines issuing from a pole P is topologized so as to be homeomorphic to the unit sphere S centered at P, under the correspondence between the Green's lines and the points of intersection with S of their tangent rays. A measure, called the Green's measure, is then introduced on $\mathcal{L}$ as proportional to the area of the corresponding subset of S, the constant of proportionality being so chosen that the total measure is 1. This is just the harmonic measure at P of the set of boundary points of $G_P>\lambda$ traced out by the Green's lines in question (λ being chosen sufficiently large). It is then shown globally that for arbitrary values λ_1, $\lambda_2>0$ the correspondence between $G_P=\lambda_1$ and $G_P=\lambda_2$ defined by the Green's lines issuing from P is a homeomorphism preserving the respective harmonic measures at P. Moreover, almost all of the Green's lines (in the sense of the Green's measure) have the property that $\inf G_P=0$ on the lines. Such Green's lines are called regular, and the set of regular Green's lines is a Borel G_δ. It is established that the finite points of $\mathcal{E}$ through which there passes no regular Green's line form a set of Lebesgue measure zero. Furthermore, for a Green's space $\Omega(\subset\mathcal{E}_\tau)$ of finite volume almost all Green's lines are rectifiable. A connection between the Green's measure and the capacity of compact sets is given, and several applications and generalizations are discussed.

Some fundamental results involving the Green's measure are concerned with functions u subharmonic on the Green's space $\mathcal{E}$. Let l be a regular Green's line issuing from P, let u_l^λ be the value of u at the point of l where $G_P=\lambda$ (>0), and set $u_l=\lim\sup_{\lambda\to 0} u_l^\lambda$. If u is bounded above, then the least harmonic majorant u^* of u satisfies the inequality

$$u^*(P)\leqq\int u_l dg(l),$$

where g is the Green's measure. Under the same hypotheses there is established the following theorem of Phragmén-Lindelöf type: if u_l is $\leqq 0$ almost everywhere on the set $\mathcal{L}'$ of regular Green's lines, then $u\leqq 0$ on $\mathcal{E}$. This leads to the "maximum principle" that the least upper bound of u on $\mathcal{E}$ is the least upper bound in Green's measure of $\{u_l: l\in\mathcal{L}'\}$. In the same order of ideas it is shown that if u is subharmonic and bounded above, then the subset of $\mathcal{L}'$ on which $u_l=-\infty$ has zero Green's measure. Application of this result to a function f analytic on a hyperbolic Riemann surface yields the theorem that if $f(z)$ tends to zero with $\lambda=G_P$ on a set of regular Green's lines having positive exterior Green's measure, then $f\equiv 0$.

The remainder of the paper deals with an extension of the Dirichlet problem containing both the ramified and geodesic cases. *M. G. Arsove* (Seattle, Wash.).

Brelot, Marcel. Principe et problème de Dirichlet dans les espaces de Green. C. R. Acad. Sci. Paris **235**, 598–600 (1952).

Let Ω be a Green's space [see the preceding review], G_P the Green's function with pole at $P(\in\Omega)$, Σ_P^λ the Green's sphere $G_P=\lambda$ (>0), $\mathcal{L}'$ the family of regular Green's lines issuing from P, $\varphi_P^\lambda(l)$ the value of φ at the point where l cuts Σ_P^λ for $l\in\mathcal{L}'$, and g_P the Green's measure relative to P. A function $\hat{\varphi}_P$ measurable-g_P on $\mathcal{L}'$ will be called a radial function for φ relative to P provided

$$\lim_{\lambda\to 0}\int|\varphi_P^\lambda-\hat{\varphi}_P|dg_P=0.$$

A function f on Ω having a continuous gradient and a finite Dirichlet integral will be designated as utilisable and assigned the norm

$$\|f\|=\left(\int \text{grad}^2 f d\mu\right)^{1/2}.$$

If $\{\Omega_k\}$ is an increasing sequence of subregions exhausting Ω and $H^f{}_{\Omega_k}$ is the solution of the Dirichlet problem on Ω_k for a utilisible function f, then $U=\lim_{k\to\infty}H^f{}_{\Omega_k}$ exists and is harmonic on Ω. Moreover, (1) U minimizes the functional $\psi(u)=\|u-f\|$ uniquely to within an additive constant over the family of all utilisable harmonic functions u; (2) $\hat{U}_P=\hat{f}_P$ almost everywhere-g_P, and $U(P)=\int\hat{f}_P dg_P$; (3) U is the only utilisable harmonic function for which $\hat{U}_P=\hat{f}_P$; (4) U is the unique utilisable function of minimum norm satisfying $\hat{U}_P=\hat{f}_P$. As the author points out, the class of utilisable functions can be substantially extended.

M. G. Arsove (Seattle, Wash.).

Brelot, Marcel. Lignes de Green et problème de Dirichlet. C. R. Acad. Sci. Paris **235**, 1595–1597 (1952).

Again Ω is taken as a Green's space and G_P the Green's function with pole at P. A harmonic function u on Ω will be called indifferent if it coincides on every region $K=D_P^\lambda$ $(G_P>\lambda)$ with the solution $H_u{}^K$ of the Dirichlet problem (assumed to exist uniquely). A subharmonic function u on Ω will be called minor if it is dominated by an indifferent harmonic function. Then $u\leqq H_u{}^K$, and as $\lambda\downarrow 0$ the functions $H_u{}^K$ converge to a harmonic function $\bar{u}$ dominating u on Ω. The function $\bar{u}$ is appropriately named the best harmonic majorant of u, since it reduces to the classical best harmonic majorant when u is subharmonic on $\bar{\Omega}$.

A function φ defined on the family $\mathcal{L}'$ of regular Green's lines is called a radial majorant of a function u if

$$\lim_{\lambda\to 0}\overline{\int}[u_l^\lambda-\varphi(l)]^+dg(l)=0,$$

where g is the Green's measure and u_l^λ is the value of u at the point of l where $G_P=\lambda$. Theorem: if a minor function u has a radial majorant φ, then $u(P)\leqq\underline{\int}\varphi dg$; if further φ is summable-g, then $\bar{u}$ has φ as a radial majorant. Applications of radial functions to Perron families and the Dirichlet problem are indicated. *M. G. Arsove* (Seattle, Wash.).

Stevenson, A. F. Note on the existence and determination of a vector potential. Quart. Appl. Math. 12, 194–198 (1954).

The author points out that the usual textbook treatment of the problem of finding a solution $\mathbf{F}$ of the equation (1) $\operatorname{curl} \mathbf{F} = \mathbf{f}$, where $\mathbf{f}$ is given, is not satisfactory if $\mathbf{f}$ is defined only in a restricted region. The usual necessary and sufficient condition, (2) $\operatorname{div} \mathbf{f} = 0$, must be supplemented by the $(n+1)$ conditions (3) $\int_{S_i} \mathbf{n} \cdot \mathbf{f}\, ds = 0$ $(i = 0, 1, 2, \cdots, n)$, where the closed surfaces S_i form the boundary of a simply connected region R. The S_i are assumed to have continuous curvature. The fact that conditions (2) and (3) are necessary and sufficient for (1) to possess a solution is proved and an example is given for which (2) is satisfied and (3) is not such that for the specified region no solution exists.

Another example is given which shows that even though both (2) and (3) are satisfied, there is no solution such that one of the components of $\mathbf{F}$ vanishes (this is usually the case). A solution is exhibited for this example.

C. G. Maple (Ames, Iowa).

Fenyő, Stefan. Über das Dirichletsche Problem bezüglich der Kugel. Publ. Math. Debrecen 3 (1953), 71–80 (1954).

On cherche à résoudre le problème en question par le potentiel d'une simple couche σ; un théorème bien connu de Picard sur les équations intégrales de première espèce permet de caractériser les données-frontiére pour lesquelles c'est possible, et de trouver un développement de σ en série de fonctions sphériques. Le cas du cercle a été traité par Picard lui-même [Rend. Circ. Mat. Palermo 29, 79–97 (1910)]; l'auteur l'attribue à J. Egerváry [Math. Phys. Lapok 23, 303–355 (1914), pp. 304–310]. *J. Deny.*

Garabedian, P. R., and Schiffer, M. On estimation of electrostatic capacity. Proc. Amer. Math. Soc. 5, 206–211 (1954).

Les auteurs montrent comment certaines formules de variation du type de Hadamard [voir à ce sujet leur article du J. Analyse Math. 2, 281–368 (1953); ces Rev. 15, 627] permettent d'obtenir aisément diverses inégalités isopérimétriques mettant en jeu des grandeurs physiques telles que la capacité électrostatique. Autre exemple d'application: la surface de capacité minima limitée par une courbe fermée C de l'espace est contenue dans l'enveloppe convexe de C. *J. Deny* (Strasbourg).

Differential Equations

Wittich, H. Zur Theorie der Riccatischen Differentialgleichung. Math. Ann. 127, 433–440 (1954).

The paper is in the main devoted to a proof of Malmquist's theorem that the Riccati equation is the only equation of the form $dw/dz = P(z, w)$ with rational $P(z, w)$ which has uniform (but not rational) solutions [cf. L. Bieberbach, Theorie der gewöhnlichen Differentialgleichungen . . . , Springer, Berlin, 1953; these Rev. 15, 703]. The theorem follows from an investigation of the properties of uniform solutions $w = f(z)$ of the equation $dw/dz = P(z, w)$ with rational $P(z, w)$ using the Nevanlinna theory of the distribution of values. Malmquist's theorem being proved, the investigation exhibits certain properties of the Riccati equation. Having reduced the equation to a representative form in which $f(z)$ is regular for large z, the argument falls into three stages. First, a system of non-overlapping circles is shown to exist, each containing a simple pole of $f(z)$ in which $f(z)$ is approximately represented by its principal part. It is then shown that $f(z)$ is of finite order and that $m(r, w) = O(\log r)$. Finally, Malmquist's theorem, some other properties of the solutions, and a few exceptional Riccati equations are obtained. *A. J. Macintyre.*

Wintner, Aurel. Remarks to an earlier note (Vol. 57, pp. 539–540). Amer. J. Math. 76, 717–720 (1954).

The author considers the differential equation

$$dy/dx = f(x, y), \quad y(0) = 0.$$

Here $f(x, y) = \sum_{k=0}^{\infty} f_k(x) y^k$ for real variable x on $0 \leqq x \leqq a + \epsilon$, $\epsilon > 0$ and complex variable y in $|y| < b$, and $|f(x, y)| \leqq M - \epsilon$. By replacing y by ty and denoting $y^t = y^t(x)$ as the solution of $dy^t/dx = f(x, ty)$, $y^t(0) = 0$, Perron [S.-B. Heidelberger Akad. Wiss. 10A, no. 8 (1919)] found $y^t(x) = \sum_{n=0}^{\infty} \varphi_n(x) t^n$ and $y(x) = \sum_{n=0}^{\infty} \varphi_n(x)$, on $0 \leqq x \leqq \min(a, \frac{1}{2} b/M)$, by formal substitution. The author improves the result of Perron by showing that the domain of convergence of the series for $y(x)$ is the "best possible" $0 \leqq x \leqq \min(a, b/M)$. In addition the series converges absolutely and uniformly and can be differentiated termwise. *L. Markus.*

Stebakov, S. A. Analysis of statically stable dynamical systems. Doklady Akad. Nauk SSSR (N.S.) 95, 455–458 (1954). (Russian)

The author considers the real, ordinary differential system

$$(1) \qquad \dot{x}_i = f_i(x_i, x_{i-1}), \quad i = 1, 2, \cdots, n, \quad x_0 \equiv x_n$$

where f_i, $\partial f_i/\partial x_i$, $\partial f_i/\partial x_{i-1} \neq 0$ are continuous in a neighborhood of the origin 0 in E^n. The system is stable if there exists a neighborhood $U(0)$ such that each solution initiating in $U(0)$ approaches 0 as $t \to \infty$. Consider the family of similar, nested parallelepipeds with sides parallel to the coordinate planes and enclosing 0. Call the collection of sides intersecting the positive x_i-axis S_{i1} and those intersecting the negative x_i-axis S_{i2}. The author remarks that if $\operatorname{sign} f_i = \operatorname{sign} (-1)^k$ whenever $(x_1, \cdots, x_n) \in S_{ik}$, then each solution curve approaches 0. Somewhat more complicated conditions of the same nature are also discussed.

L. Markus (New Haven, Conn.).

Kolmogorov, A. N. On dynamical systems with an integral invariant on the torus. Doklady Akad. Nauk SSSR (N.S.) 93, 763–766 (1953). (Russian)

The author considers a dynamical system defined on a 2-dimensional torus T^2 by the system of differential equations

$$(1) \qquad \frac{dx}{dt} = A(x, y), \quad \frac{dy}{dt} = B(x, y),$$

and possessing an invariant integral $I(g) = \iint_g U(x, y)\, dx\, dy$, where A, B and U are univalued, analytic periodic functions of x and y with period 2π. Here x and y are real coordinates mod 2π, $A^2 + B^2 > 0$, $U > 0$ on the whole of T^2. It is then known [Nemyckiĭ and Stepanov, Qualitative theory of differential equations, 2nd. ed., Gostehizdat, Moscow-Leningrad, 1949; for a review of the 1st ed. see these Rev. 10, 612] that there exists an analytic transformation of coordinates which transforms the system (1) into the system

$$(2) \qquad \frac{dx}{dt} = \frac{1}{F(x, y)}, \quad \frac{dy}{dt} = \frac{\gamma}{F(x, y)}$$

with an integral invariant $I(g)=\int\int_g F(x,y)dxdy$ where γ is a constant.

The following theorem is asserted. Theorem 1. If there exist constants $c>0$ and $h>0$ such that for all positive integers m and n

$$\text{(i)} \qquad |m-n\gamma|\geq ch^n,$$

then there exists an analytic transformation of coordinates which transforms the system (2) into the system

$$\text{(3)} \qquad \frac{du}{dt}=\lambda_1, \quad \frac{dv}{dt}=\lambda_2,$$

where λ_1, λ_2 are constants and $\lambda_2=\gamma\lambda_1$ and with the integral invariant $I(g)=K\int\int_g dudv$. Condition (i) is fulfilled for every γ except for a set of Lebesgue measure zero (c and h depend on γ). It follows that system (1) has a pure point spectrum with analytic proper functions.

For those irrational numbers which do satisfy (i) the author states: Theorem 2. Each of the following conditions is possible for a suitable choice of γ and $F(x,y)$: The system (2) can be transformed into (3) by (I) an infinitely differentiable but not analytic transformation, (II) a k-differentiable but not $(k+1)$-differentiable transformation, (III) an everywhere-discontinuous transformation; and (IV) the system (2) cannot be transformed into (3) at all. In (I), (II) and (III) the original system (1) has a pure point spectrum but the proper functions are respectively not analytic, not $k+1$ differentiable and everywhere discontinuous. The conjecture is made that in (IV) the spectrum is necessarily continuous but only a considerably weaker result is proved. In all statements related to Theorem 2 the notions of analyticity, differentiability, etc. are interpreted modulo sets of Lebesgue measure zero. The method of obtaining the system (3) from (2) is obtained and discussed.

Y. N. Dowker (London).

Grabar', M. I. On strong ergodicity of dynamical systems. Doklady Akad. Nauk SSSR (N.S.) 95, 9–12 (1954). (Russian)

A dynamical system is called strongly ergodic if it has only one invariant, normalized measure. A dynamical system is called irreducible if every trajectory is everywhere dense in R.

The author gives an example of a dynamical system defined on the 3-dimensional torus T^3 by a system of differential equations which is irreducible but not strongly ergodic. Thus, while examples of dynamical systems which are irreducible but not strongly ergodic have been given previously [e.g., Oxtoby, Bull. Amer. Math. Soc. 58, 116–136 (1952); these Rev. 13, 850], this seems to be the first such example which is not abstractly constructed and which is given on a manifold. More precisely, the following theorem is asserted: Let (i) $dx/dt=\alpha_1$, $dy/dt=\alpha_2$, $dz/dt=\lambda/\phi(x,y)$, where x, y and z are real coordinates on T^3, considered mod 2π, $\phi(x,y)$ is continuous, positive and periodic mod 2π, $\alpha_2/\alpha_1=\gamma$ is irrational and $\lambda\neq0$ is an arbitrary real number; then it is possible to choose α_1, α_2 and an analytic $\phi(x,y)$ such that the dynamical system defined by (i) is irreducible for every $\lambda\neq0$. And if (A) denotes the system

$$\frac{dx}{dt}=\alpha_1\phi(x,y), \quad \frac{dy}{dt}=\alpha_2\phi(x,y),$$

then (1) if $k\lambda$ is not a proper value of (A) for any $k\neq0$, then the system (i) is strongly ergodic; (2) if for some integer $k\neq0$, $k\lambda$ is a proper value of (A), then the system (i) is not strongly ergodic. Both (1) and (2) can be realized.

The proof is outlined in some detail and depends on lemmas concerning general continuous and discrete dynamical systems. The proof of the possibility of realizing (2) depends also on results of the paper reviewed above.

Y. N. Dowker (London).

Marx, Imanuel. On the structure of recurrence relations. Michigan Math. J. 2, 45–50 (1954).

It is assumed that the characteristic functions associated with a second-order linear differential equation satisfy a pair of recurrence relations of the type which are well known for many special functions. Necessary conditions which must be satisfied by the coefficients in these relations are derived in a variety of forms. The author states that his results "may be of assistance in determining coefficients for recurrence relations not yet completely known."

T. E. Hull (Vancouver, B. C.).

Carrier, G. F. Boundary layer problems in applied mathematics. Comm. Pure Appl. Math. 7, 11–17 (1954).

Two techniques for solving boundary-layer problems are compared, one due to the author [Advances in applied mechanics, v. 3, Academic Press, New York, 1953, pp. 1–19; these Rev. 15, 959] and one due to M. J. Lighthill [Philos. Mag. (7) 40, 1179–1201 (1949); these Rev. 11, 518]. Such problems occur when the most highly differentiated terms in differential equations involve small nonlinearities. The small nonlinearities may be unimportant except near boundaries. The author's method is to piece together relatively simple solutions having limited ranges of validity, and which are found by various changes of scale or "stretching" transformations involving powers of a small parameter. In Lighthill's method, on the other hand, the dependent variable w and the independent variable x are represented parametrically in terms of a new variable z in the following way:

$$\text{(1)} \qquad w=w_0(z)+w_1(z)+\cdots,$$

$$\text{(2)} \qquad x=z+x_1(z)+\cdots.$$

$x_1(z)$, $x_2(z)$, $\cdots$ are taken to be singular at $z=0$ so that $x=0$ does not correspond to $z=0$. Subject to this singularity condition there is a wide choice of functions $x_i(z)$; their selection may be made in the interests of simplicity. Suppose the boundary is at $x=0$, and that $w_0(x)$ is the solution to the equation when the nonlinearity is neglected. Then (2) may be solved for z and the result substituted into (1) to give an expression which is valid even when $w_0(x)$ is not a good approximation.

The author solves two problems by both methods by way of comparison, and notes that his method works for van der Pol's equation, while Lighthill's does not. He concludes that each method had advantages in particular circumstances, and that the two can be used profitably together.

E. Pinney (Berkeley, Calif.).

Basch, A. Über Schwingungen von Systemen mit zwei Freiheitsgraden. Österreich. Ing.-Arch. 8, 83–86 (1954).

Stability about the rest position of a two-degree-of-freedom system is investigated by the usual perturbation procedure. The system is assumed conservative, and weak stability is shown to follow.

E. Pinney.

Zlámal, Miloš. Über ein Kriterium der Stabilität von Liapounoff. Čehoslovack. Mat. Ž. 3(78), 257–264 (1953). (Russian. German summary)

The differential equation $y''+p(x)y=0$ is considered where $p(x)$ is a real continuous periodic function of period π. The author proves the solutions are bounded if for some real $a>0$

$$\int_0^\pi (p(x)-a)^2 dx < 2a \min [|\sin \tfrac{1}{2}a\pi|, |\cos \tfrac{1}{2}a\pi|].$$

Also sufficient for boundedness when $p(x) \geqq a^2 > 0$ is

$$\int_0^\pi p(x)dx < \pi a^2 + 2af(a),$$

where $f(a)=|\cos \tfrac{1}{2}a\pi|$ if $2k \leqq a < 2k+1$ and $f(a)=|\sin \tfrac{1}{2}a\pi|$ if $2k+1 \leqq a < 2k+2$ where k is an integer. *N. Levinson.*

Mizohata, Sigeru. Sur certaines équations différentielles régissant quelques phénomènes héréditaires. Math. Japonicae 3, 1–5 (1953).

Periodic solutions for a certain nonlinear functional equation are considered by a method similar to Minorsky's "stroboscopic" method. *E. Pinney* (Berkeley, Calif.).

Mazzoni, Pacifico. Equazioni differenziali per le rendite continue. Giorn. Ist. Ital. Attuari 15, 85–92 (1952).

The continuous annuity certain regarded as function of the force of interest satisfies various differential equations, e.g., $xy'+(1-nx)y-n=0$. During the evaluation limits are found for combinations of the derivatives used in interpolation formulae. *P. Johansen* (Copenhagen).

Nitsche, Johannes. Über Systeme kanonischer Differentialgleichungen und das zugehörige singuläre Eigenwertproblem. Wissensch. Z. Univ. Leipzig. Math.-Nat. Reihe 1952/1953, 193–226 (1953).

The first portion of the paper is concerned with criteria for a matrix differential equation $U'(z)+A(z)U(z)=0$ to have a regular singular point at $z=0$. The major part of the paper deals with singular boundary problems for a canonical system of the form

$$(*) \qquad \mathfrak{J}u'+B(x)u=\lambda C(x)u,$$

where $\mathfrak{J}=\begin{pmatrix} 0 & e \\ -e & 0 \end{pmatrix}$ with e the nth order identity matrix, $B(x)$ and $C(x)$ are symmetric matrices of order $2n$ with real-valued continuous elements on $l_1<x<l_2$, all elements of C are zero except those belonging to a particular rth order principal minor that is positive definite, and (*) is normal in the sense that if u satisfies (*) and $Cu\equiv 0$ on $l_1<x<l_2$ then $u\equiv 0$. In terms of a metric introduced by means of the semi-definite inner product $(u,v)=\int_{l_1}^{l_2} v^*Cu\,dx$, there are established analogues of the Weyl limit circle and limit-point cases for a differential equation of the second order. Discussed examples include systems (*) with periodic coefficients, and a real self-adjoint equation $\sum_{\nu=0}^{n} (-1)^\nu(\beta_\nu y^{(\nu)})^{(\nu)}=\lambda\gamma_0 y$ possessing a regular singular point at the endpoint $l_1=0$. For the instance when the given analogue of the limit circle case holds for both l_1 and l_2 the author discusses the determination of boundary conditions and the reduction of the boundary problem to an integral equation; particular attention is given to the case when the interval of consideration is $0<x\leqq l$ and $x=0$ is a regular singular point. Also discussed is the reduction to an integral equation of the boundary problem arising when for each endpoint the system (*) possesses for $\lambda=i$ exactly n linearly independent solutions of integrable square, in the sense of the metric mentioned above, on a neighborhood of the endpoint. The author was evidently unaware of a paper by Kodaira [Amer. J. Math. 72, 502–544 (1950); these Rev. 12, 103] treating the spectral and expansion theory of singular boundary problems for real formally self-adjoint differential operators of even order. *W. T. Reid* (Evanston, Ill.).

Zlámal, Miloš. Über eine Eigenwertaufgabe bei der Differentialgleichung $y^{(n)}+\lambda A(x)y=0$. Publ. Fac. Sci. Univ. Masaryk 1953, 91–99 (1953). (Russian summary)

For the boundary problem

$$y^{(n)}+\lambda A(x)y=0 \quad (n\geqq 2),$$
$$y(b)=0, \quad y^{(\alpha)}(a)=0 \quad (\alpha=0,1,\cdots,k-1,k+1,\cdots,n-1),$$

where $A(x)$ is a positive continuous function on $a\leqq x\leqq b$, the author establishes by Sturmian methods the existence of infinitely many positive simple proper values $0<\lambda_0<\lambda_1<\cdots$ with the proper function $y_\nu(x)$ corresponding to λ_ν possessing exactly ν zeros on $a<x<b$. In particular, for $k=n-1$ the considered problem reduces to that treated by G. Sansone [Ann. Mat. Pura Appl. (4) 24, 209–236 (1945); these Rev. 9, 36] using the method of integral equations. The above results are extended to a class of problems involving the same end conditions and a more general differential equation of the form $y^{(n)}+Q(x,\lambda)y=0$. *W. T. Reid.*

Šnol', I. È. On the behavior of eigenfunctions. Doklady Akad. Nauk SSSR (N.S.) 94, 389–392 (1954). (Russian)

Let $Ly=-y''+q(x)y$. Let $\phi(x,\lambda)$ satisfy $L\phi=\lambda\phi$ for $0<x<\infty$, $\phi(0,\lambda)=1$, $\alpha\phi(0,\lambda)+\beta\phi'(0,\lambda)=0$. Let $\rho(\lambda)$ be the spectral function associated with the expansion theorem for this problem. The author proves that if $q>0$ (or $>$constant) then for almost all λ (with respect to measure by $\rho(\lambda)$) and for any $\epsilon>0$, $|\phi(x,\lambda)|<c(\lambda,\epsilon)x^{1+\epsilon}$. Other results are given. *N. Levinson* (Cambridge, Mass.).

Zimmerberg, Hyman J. On fundamental matrix solutions. Proc. Amer. Math. Soc. 5, 391–394 (1954).

L'autore dimostra che, assegnato comunque un sistema di equazioni differenziali lineari omogenee del primo ordine, è sempre possibile costruire un sistema fondamentale di integrali a partire dalla matrice di Green relativa a un arbitrario sistema di condizioni ai limiti. Ciò anche quando, ammettendo il problema ai limiti considerato soluzioni non nulle, la detta matrice di Green vale in senso generalizzato. *C. Miranda* (Napoli).

Krein, M. On a method of effective solution of an inverse boundary problem. Doklady Akad. Nauk SSSR (N.S.) 94, 987–990 (1954). (Russian)

Referring to an earlier paper [same Doklady (N.S.) 93, 617–620 (1953); these Rev. 15, 796] the author introduces

$$\Phi(t)=\int_0^\infty \frac{1-\cos\lambda^{1/2}t}{\lambda}d\tau(\lambda)$$

where τ is the spectral function. Necessary and sufficient conditions on Φ are given for Φ to be represented by some spectral function as above. Also a homogeneous integral equation with kernel $\Phi''(|t-s|)$ and depending on a parameter is given, the solution of which leads to the determination of the $M(x)$ that occurs in the operator of which τ is the spectral function. Reference to practical applications is made. *N. Levinson* (Cambridge, Mass.).

Coddington, E. A. The spectral matrix and Green's function for singular self-adjoint boundary value problems. Canadian J. Math. 6, 169–185 (1954).

This paper is a continuation of earlier work of the author [Proc. Nat. Acad. Sci. U. S. A. **38**, 732–737 (1952); these Rev. **14**, 278] on a formally self-adjoint differential operator $L=p_0(d/dx)^n+p_1(d/dx)^{n-1}+\cdots+p_n$, where the coefficients p_k are complex-valued functions with $n-k$ continuous derivatives on an open real interval (a, b) on which $p_0(x)\neq 0$. For the set of Green's functions $\{G_\delta\}$ associated with self-adjoint boundary problems on closed bounded subintervals δ of (a, b) certain properties of compactness are established, and with the aid of these results and a simple limiting process the existence and uniqueness of the Green's function and the spectral matrix are established in the following two cases: (i) no boundary conditions at a and b are required to obtain a self-adjoint boundary problem; (ii) the endpoint a is finite, (a, b) can be replaced by $[a, b)$, and a self-adjoint boundary problem results by imposing conditions at a alone. The formula relating Green's function to the spectral matrix is established by a type of argument used by Levinson [Duke Math. J. **18**, 57–71 (1951); these Rev. **12**, 828] for the case $n=2$, p_k real. *W. T. Reid* (Evanston, Ill.).

Partial Differential Equations

have

✶Courant, R. On the classification of partial differential equations. Scientific papers presented to Max Born, pp. 29–32. Hafner Publishing Co. Inc., New York, N. Y., 1953. $2.50.

This is an expository article in which the author shows how a semilinear system of partial differential equations which may be written: $U_y=AU_x+B$, in matrix notation, may be transformed to the form $V_y=A'V_x+B'$, where A' has the Jordan normal form, and how the "boxes", i.e. the submatrices along the main diagonal in the normal form, may be used to classify the equation. *D. L. Bernstein.*

Birkhoff, Garrett. Classification of partial differential equations. J. Soc. Indust. Appl. Math. 2, 57–67 (1954).

Considérons le système d'équations aux dérivées partielles à coefficients constants (*) $\partial u/\partial t=P(\partial/\partial y_1, \cdots, \partial/\partial y_n)u$, $u=(u_1, \cdots, u_n)$, les éléments de la matrice P étant des polynômes. Appelons polynôme de stabilité du système (*) l'expression $\sigma(\lambda, q)=\mathrm{dtm}\,[\lambda E-P(q)]$, $q=(q_1, \cdots, q_n)$. Le système (*) est dit stable si, pour toute valeur réelle de q, toutes les racines $\lambda(q)$ du polynôme de stabilité ont leur partie réelle négative, régulier, si ces parties réelles admettent une borne supérieure finie indépendante de q. L'auteur affirme que pour le système régulier, le problème des valeurs initiales est bien posé dans un certain espace de Banach qui contient tous les polynômes trigonométriques. Si le système est stable, c'est seulement le problème avec second membre, la solution comme le second membre étant identiquement nulle pour $t<0$, qui est résoluble. L'auteur indique en outre que les conditions de stabilité et de régularité se ramènent à constater qu'un certain nombre de polynômes sont identiquement positifs. Il examine enfin les relations entre la régularité et l'hyperbolicité, rappelant dans le cas du système (*) un certain nombre de théorèmes dûs à L. Gårding [Acta Math. **85**, 1–62 (1951); ces Rev. **12**, 831].

Il s'agit d'un article d'information générale; on n'y trouve ni précisions, ni démonstrations. *H. G. Garnir* (Liège).

Leray, Jean. The physical facts and the differential equations. Proceedings of the symposium on special topics in applied mathematics, Northwestern University, 1953. Amer. Math. Monthly 61, no. 7, part II, 5–7 (1954).

Szarski, J. Systèmes d'inégalités différentielles aux dérivées partielles du premier ordre, et leurs applications. Ann. Polon. Math. 1, 149–165 (1954).

Dans un travail précédent [Ann. Soc. Polon. Math. **24**, no. 2, 9–16 (1954); ces Rev. **15**, 626], l'auteur a montré l'existence et l'unicité de la solution du "problème de Cauchy" pour un système et des données satisfaisant à des conditions générales de régularité (existence de dérivées continues bornées, jusqu'au 3ème ordre), de la forme

$$\frac{\partial u_1}{\partial x}=f_1\left(x, y, u_1, u_2, \frac{\partial u_1}{\partial y}\right),\quad \frac{\partial u_2}{\partial x}=f_2\left(x, y, u_1, u_2, \frac{\partial u_2}{\partial y}\right),$$

et pour ceux de forme analogue générale, les u étant en nombre quelconque m, les variables y en nombre quelconque n. Dans un autre travail [ibid. **21**, 7–25 (1948); ces Rev. **10**, 195], il avait considéré les systèmes d'inégalités obtenues en remplaçant le signe = par le signe < dans toutes les relations du système de la forme générale indiquée; il avait démontré que, sous certaines hypothèses simples, les inégalités

$$u_\mu(x, y_1, \cdots, y_n)<v_\mu(x, y_1, \cdots, y_n)$$

supposées vraies pour x_0 restent vraies dans tout un domaine fixé d'avance, et il avait appliqué ce résultat aux solutions du système général d'équations indiqué plus haut.

La présente note est consacrée à des généralisations au cas où, au lieu d'une seule variable tenant le rôle de x, il y en a un nombre quelconque k. On arrive finalement au résultat suivant: Soit le système d'équations

$$\frac{\partial u_\mu}{\partial x_\nu}=g_\nu^\mu\left(x_1, \cdots, x_k, y_1, \cdots, y_n, u_1, \cdots, u_m, \frac{\partial u_\mu}{\partial y_1}, \cdots, \frac{\partial u_\mu}{\partial y_n}\right)$$
$$(\mu=1, 2, \cdots, m,\ \nu=1, 2, \cdots, k)$$

où les g et les données initiales (valeurs de u pour x_1^0, $x_2^0, \cdots, x_k^0$) satisfont à des conditions générales de régularité; sous certaines hypothèses simples, les inégalites

$$u_\mu(x_1, \cdots, x_k, y_1, \cdots, y_n)\leqq v_\mu(x_1, \cdots, x_k, y_1, \cdots, y_n)$$

supposées vraies pour le système initial, $x_1^0, x_2^0, \cdots, x_k^0$, subsistent dans tout un domaine fixé d'avance. *M. Janet* (Paris).

Pailloux, Henri. Equations qui se décomposent. C. R. Acad. Sci. Paris 237, 960–961 (1953).

This note gives two ways of writing general solutions of:

$$r(T, X_i)u(t, x_i)=0 \quad \left(T=\frac{\partial}{\partial t}, X_i=\frac{\partial}{\partial x_i}, i=1, 2, \cdots, n\right),$$

where $r(b, a_i)$ is an entire rational function of all its variables with constant coefficients. The author then shows how this can be used to obtain information about solutions of equations of the form $r^m u=0$, or $r_1r_2u=0$, and other results of similar nature. No proofs are given. *D. L. Bernstein.*

✶Bouligand, G. Sur une classe d'équations aux dérivées partielles du premier ordre. Convegno Internazionale di Geometria Differenziale, Italia, 1953, pp. 297–299. Edizioni Cremonese, Roma, 1954. 4000 Lire. on order

Ce résumé de conférence reproduit en substance une note antérieure de l'auteur aux C. R. Acad. Sci. Paris **237**, 772–774 (1953); ces Rev. **15**, 345. *Chr. Pauc* (Nantes).

Geiringer, Hilda. Bemerkung zur Theorie der Charakteristiken. Österreich. Ing.-Arch. 8, 107–109 (1954).

For a system of two first-order partial differential equations in two independent variables the author gives a brief method for the reduction to characteristic form.

M. H. Protter (Berkeley, Calif.).

Lukomskaya, M. A. Solution of some systems of partial differential equations by means of inclusion in a cycle. Akad. Nauk SSSR. Prikl. Mat. Meh. 17, 745–747 (1953). (Russian)

The theory of Σ-differentiation and Σ-integration for systems of the form

$$c_1u_x+d_1v_x=a_1u_y+b_1v_y,$$
$$c_2u_x+d_2v_x=a_2u_y+b_2v_y$$

is a special case of the theory of pseudo-analytic functions developed by Bers [Theory of pseudo-analytic functions, Inst. Math.-Mech., New York Univ., 1953; these Rev. 15, 211]. In this note certain equations of the above form which arise in problems in mechanics are shown to be included in a 1-cycle (period 1 in the terminology of Bers). In this manner solutions can be obtained in terms of generalized powers which in turn may be found in finite form. In the final paragraph it is shown that the system $pu_x=v_y$, $-pu_y=v_x$, $p=p(x, y)$ harmonic, has period 1. *M. H. Protter.*

Pucci, Carlo. Il problema di Cauchy per le equazioni lineari a derivate parziali. Ann. Mat. Pura Appl. (4) 35, 129–153 (1953).

The author establishes the existence and uniqueness of a solution $u(x_1, \cdots, x_r, t)$ of the differential equation of order m:

$$\sum_{i_1+\cdots+i_r+s\leq m} a_{i_1,\cdots,i_r,s}(t)\frac{\partial^{i_1+\cdots+i_r+s}u}{\partial x_1^{i_1}\cdots\partial x_r^{i_r}\partial t^s}=f(x_1, \cdots, x_r, t)$$

with the initial conditions:

$$u_{t^i}(x_1, x_2, \cdots, x_r, 0)=u_i(x_1, \cdots, x_r) \quad (i=0, 1, \cdots, m-1).$$

Here the coefficients in the equation are assumed to have bounded derivatives of all orders in $I=[-h, h]$ and the functions f and u_i belong to a certain class $\Gamma_{\sigma(h)}$ of functions, where Γ_σ is the class of functions which, for all $t\in I$ and $(x_1, \cdots, x_r)\in A$, have partial derivatives with respect to the x_i's of all orders, and where these derivatives have bounds of the form:

$$\left|\frac{\partial^{i_1+\cdots+i_r}\varphi}{\partial x_1^{i_1}\cdots\partial x_r^{i_r}}\right|<M_\varphi\frac{[i_1+\cdots+i_r]!}{\rho^{i_1+\cdots+i_r}} \quad (\rho>\sigma)$$

in $C_h=A-I$. Then he establishes the existence of a unique solution of class $G_{\sigma(h)}$ in the region C_h. (Here G_σ is the subclass of Γ_σ having the additional property that all the derivatives occurring in the differential equation are continuous in C_h.) He also exhibits a series form of the solution, and proves the convergence of the series is uniform. The second theorem of the paper generalizes the results to somewhat broader classes than G_σ and Γ_σ. In the last theorem certain conditions on the given data are shown to be sufficient for the series solution to have only a finite number of non-zero terms. *D. L. Bernstein* (Rochester, N. Y.).

Mihlin, S. G. Variational methods of solution of problems of mathematical physics. Uspehi Matem. Nauk (N.S.) 5, no. 6(40), 3–51 (1950). (Russian)

The present paper is intended as a survey of the subject matter indicated in the title. Only linear boundary-value problems and their associated minimum problems for quadratic functionals are considered. Proofs of most of the results are given. Section 1 is a brief historical outline. The general problem consists in the determination of u satisfying

$$(*) \qquad Au=f,$$

where f is a given vector in a complex Hilbert space, and A is a linear (additive plus homogeneous) positive (i.e. $(Av, v)>0$ for $v\neq 0$) operator defined on a linear subset D_A dense in the Hilbert space in question. Section 2 contains the theorem that the determination of u in D_A satisfying (*) is equivalent to finding the minimum of the quadratic functional

$$(**) \qquad F(u)=(Au, u)-(u, f)-(f, u)=(Au, u)-2\mathrm{Re}(u, f).$$

Section 3 considers the minimum problem for $F(u)$ of (**), under the additional hypothesis that A is positive definite, i.e. there exists $\gamma>0$ such that $(Av, v)\geqq\gamma^2(v, v)$, for any v in D_A. Proofs are given of the theorems of K. O. Friedrichs [Math. Ann. 109, 465–487, 685–713 (1934)] that under these hypotheses the Hilbert space H_0 obtained by (Cauchy) completing D_A, when D_A is endowed with the scalar product $[\ ,\ ]$ defined by $[u, v]=(Au, v)$, for u, v in D_A, may be considered as a subspace of the original Hilbert space H; and that a positive definite operator defined on a dense subset of H may be extended to a self-adjoint operator. Section 4 contains remarks about the minimization of $F(u)$ when (u, f) is replaced by a linear functional lu, not necessarily bounded; and when the operator A in $F(u)$ is assumed to be just positive, and not positive definite. Section 5 contains a proof of the convergence of Ritz' method for the solution of (*), when A is positive definite. The method consists in forming the sequence

$$u_n=\sum_{k=1}^n a_k\varphi_k, \quad n=1, 2, \cdots,$$

where the chosen vectors $\varphi_1, \varphi_2, \cdots$ constitute a complete linearly independent sequence in H_0, and the coefficients a_k are determined from the set of linear equations

$$\sum_{k=1}^n [\varphi_k, \varphi_m]a_k=(f, \varphi_m), \quad m=1, 2, \cdots, n.$$

If the Ritz "cöordinate vectors" φ_i are chosen from D_A itself, rather than just from H_0, the system of equations for the a_k may be written

$$(***) \qquad \sum_{k=1}^n (A\varphi_k, \varphi_m)a_k=(f, \varphi_m), \quad m=1, 2, \cdots, n,$$

which is B. G. Galerkin's [Vestnik Inženerov no. 19, 897–908 (1915)] form of the Ritz equations.

Section 6 is devoted to a second variational problem (containing as a special case Dirichlet's principle for the solution of the Dirichlet problem for the homogeneous Laplace equation), whose solution is given by the method of orthogonal projection [H. Weyl, Duke Math. J. 7, 411–444 (1940); these Rev. 2, 202; M. I. Višik, Mat. Sbornik N.S. 25(67), 189–234 (1949); these Rev. 11, 520; O. Nikodym, Mathematica, Cluj 9, 110–128 (1935)]. Section 7 is concerned with the application of the variational method to the determination of eigenvalues of positive definite A. There follow sections showing the applicability of the variational method to certain linear boundary-value problems (nonhomogeneous equation with homogeneous boundary conditions) and also eigenvalue problems. By "application" is meant the following: usually the operator occurring in the

equation can be thought of as acting on a subset of $L_2(\Omega)$, where Ω is the domain where the differential equation holds, and in the boundary-value problems considered it is shown that this operator (call it A) is positive definite, while in the eigenvalue problems it is shown that every bounded set (in the space H_0 corresponding to A) is compact in $L_2(\Omega)$, so that the previously proved results apply.

Section 9 deals with the homogeneous Dirichlet problem for the elliptic equation

$$Au=-\sum_{\alpha,\beta=1}^{m}\frac{\partial}{\partial x_\alpha}\left(A_{\alpha\beta}(P)\frac{\partial u}{\partial x_\beta}\right)+B(P)u=f(P),\quad A_{ik}=\bar{A}_{ki},$$

while section 11 deals with the first boundary-value problem of the theory of elasticity [Friedrichs, Ann. of Math. (2) **48**, 441–471 (1947); these Rev. **9**, 255]. Section 12 discusses the "imbedding" theorems of S. L. Sobolev [Mat. Sbornik N.S. 2(44), 465–499 (1937); 4(46), 471–497 (1938)], while the two succeeding sections discuss its application to Neumann's problem and to boundary-value problems with degenerate boundaries (with dimension less than $m-1$, if m is the dimension of the space). Section 16 discusses the sense in which the solution of the variational problem satisfies the differential equation and the boundary conditions. Section 17 deals with the "method of least squares" for the solution of (*). This method consists in choosing "coordinate functions" $\varphi_1, \varphi_2, \cdots$, and finding for each $n=1, 2, \cdots$ that linear combination $u_n=\sum_{k=1}^{n}a_k\varphi_k$ which minimizes $\|Au-f\|^2$. The a_k satisfy the system of linear equations

$$\sum_{k=1}^{n}a_k(A\varphi_k, A\varphi_m)=(f, A\varphi_m),\quad m=1, 2, \cdots, n.$$

Sufficient conditions for the convergence of the sequence u_n to a solution of (*) are given. Section 18 shows the connection between the method of least squares and the variational method of minimizing (**). Specifically, if A is self-adjoint and positive-definite, then the variational method of solving $Au=f$ by minimizing $F(u)$ of (**) is equivalent to the application of the method of least squares to the equation $A^{\frac{1}{2}}u=A^{-\frac{1}{2}}f$. The remaining sections deal essentially with the convergence of Galerkin's method (see (***)), its application to eigenvalues, and its generalization due to G. I. Petrov [Akad. Nauk SSSR. Prikl. Mat. Meh. (N.S.) 4, no. 3, 3–12 (1940)], which consists in the following: in seeking a solution of (*), choose two "coordinate" sequences φ_n and ψ_n, and construct approximate solutions to (*) of the form $u_n=\sum_{k=1}^{n}a_k\psi_k$, where the constants a_k are determined by requiring that Au_n-f be orthogonal to $\varphi_1, \cdots, \varphi_n$.

J. B. Dias (College Park, Md.).

✱Mihlin, S. G. Pryamye metody matematičeskoĭ fizike. [**Direct methods in mathematical physics**]. Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1950. 428 pp.

The author understands, by the phrase "direct methods", methods for the approximate solution of problems of differential and integral equations which involve the replacement of the original problem by the solution of systems of algebraic equations. The present book, addressed to mathematical engineers, is devoted to a development of the following four main methods of this general description: the method of Ritz, the method of Galerkin, the method of least squares, and the method of finite differences. The principal aim is to ascertain the kind of approximation to the exact solution which is furnished by each of these procedures. The first three methods furnish, generally speaking, approximations which converge "in the mean" to the exact solution. Chapter I is devoted to an introduction to Hilbert space and Lebesgue integration. Chapters II, III, and IV deal with the Ritz method and variational methods [see, in this connection, the preceding review for a more extended description of these questions]. Chapter V is devoted to the method of Galerkin and chapter VI to the method of least squares, which are also described in detail in the review just mentioned. The level of exposition is very high, and the readability of these chapters is facilitated by the concise summaries placed at the end of chapters III, V and VI. Chapter VII treats the method of finite differences, with particular reference to the "method of lines" [see L. V. Kantorovich and V. I. Krylov, Approximate methods in higher analysis, 2nd ed., Gostehizdat, Leningrad-Moscow, 1941] which, in essence, consists in the approximate integration of problems in partial differential equations by replacing them by the integration of systems of ordinary differential equations. Throughout the book there are numerous concrete examples of numerical applications of these methods, completely worked out. [In connection with the discussion of the methods of Rayleigh-Ritz and Trefftz, the reviewer would like to mention the unified discussion given by the reviewer and A. Weinstein [J. Math. Physics **26**, 133–136 (1947); these Rev. **9**, 211] and by the reviewer [Proc. Symposium on Spectral Theory and Differential Problems, Oklahoma Agric. and Mech. Coll., Stillwater, Okla., 1951, pp. 279–289; these Rev. **13**, 235]; while in connection with the determination of eigenvalues, mention should be made of the method of A. Weinstein [Mémor. Sci. Math., no. 88, Gauthier-Villars, Paris, 1937] (which complements the Rayleigh-Ritz method); see also N. Aronszajn [Proc. Nat. Acad. Sci. U. S. A. **34**, 474–480, 594–601 (1948); these Rev. **10**, 382] and H. Weyl [Bull. Amer. Math. Soc. **56**, 115–139 (1950); these Rev. **11**, 666].]

J. B. Dias (College Park, Md.).

✱Mihlin, S. G. Problema minimuma kvadratičnogo funkcionala. [**The problem of the minimum of a quadratic functional.**] Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1952. 216 pp. 6.15 rubles.

On order

The present book consists of a careful presentation, with more regard given to theoretical questions, of the general subject matter which is surveyed in the paper reviewed second above and which was presented to the mathematical-engineering audience in the book reviewed above. For a more detailed description of these questions, reference is made to the first cited review. Chapter I, entitled "Formulation and solution of variational problems", is concerned with the solution of the "fundamental variational problems", Friedrichs' extension of a positive definite operator, and the methods of Galerkin and of least squares. Chapter II deals with various auxiliary considerations, for example, Sobolev's integral identity [Some applications of functional analysis to mathematical physics, Leningrad. Gos. Univ., 1950; these Rev. **14**, 565] and mean-value theorems for linear elliptic partial differential equations of the second order [Mihlin, Doklady Akad. Nauk (N.S.) **77**, 377–380 (1951); these Rev. **12**, 830]. Chapter III contains the application of the variational methods to boundary-value problems for equations of elliptic type and chapter IV the specific application of the same methods to the theory of elasticity.

J. B. Dias (College Park, Md.).

Serrin, J. B. On the Phragmén-Lindelöf principle for elliptic differential equations. J. Rational Mech. Anal. 3, 395–413 (1954).

The author deals with the Phragmén-Lindelöf principle for differential equations

$$L[u]=au_{xx}+2bu_{xy}+cu_{yy}+du_x+eu_y=0$$

which are uniformly elliptic in $y>0$ and for which there exists a decreasing function $p(r)$ with convergent integral over $\langle 0, \infty\rangle$ such that $(d^2+e^2)^{1/2}\leqq p(r)(a+c)$ for $x^2+y^2=r^2$. Let d be a subregion of $y>0$ with boundary $\bar{D}$ in a strip $0\leqq y\leqq\sigma$; let $u(x, y)$ and $v(x, y)$ satisfy the conditions $L[u]\geqq 0$ in D, $u\leqq 0$ in $\bar{D}$ and $L[v]\leqq 0$ in D, $v\geqq 0$ in D. If we denote $M(r)=\max u$ (for $x^2+y^2=r^2$) and $m(Y)=\min v$ (for $y=Y$), it is proved that $\alpha=\lim_{r\to\infty} M(r)/r$ and $\beta=\lim_{Y\to\infty} m(Y)/Y$ exist; it is shown that α is non-negative and that β is finite. Throughout D the inequality $\beta u\leqq\alpha v$ holds and equality at one point in D implies $u=\alpha\psi$, $v=\beta\psi$ where $\psi(x, y)$ satisfies the conditions $L[\psi]=0$, $\psi=0$ in $\bar{D}$, $\lim_{y\to\infty}\psi/y=1$ and $\psi<k(\Sigma)$ in any strip $0\leqq y\leqq\Sigma$. This main result is an extension of previous work by Gilbarg [same J. 1, 411–417 (1952); these Rev. 14, 279] and Hopf [ibid. 1, 419–424 (1952); these Rev. 14, 279], and is derived from a more special result of Gilbarg by proper use of Hopf's formulation and derivation of the maximum principle for elliptic differential equations [S.-B. Preuss. Akad. Wiss. 1927, 147–152]. Finally, the author gives an extension of the equation

$$au_{xx}+2bu_{xy}+cu_{yy}+du_x+(e-mcy^{-1})u_y=0.$$

M. Schiffer (Stanford, Calif.).

Linhart, J. G. The radiation Green's functions. J. Franklin Inst. 258, 99–112 (1954).

Müller, Claus. On the behavior of the solutions of the differential equation $\Delta U=F(x, U)$ in the neighborhood of a point. Comm. Pure Appl. Math. 7, 505–515 (1954).

The purpose of the author is to investigate the local behaviour of the solutions of the above equation and in particular of the equation

$$(1)\qquad \Delta U+k^2(x)U=0.$$

Δ denotes the p-dimensional Laplacian, (x) is the spatial vector in p Euclidean dimensions, $k^2(x)$ is a (not necessarily analytic) function. The length of the vector (x) is denoted by $|x|$.

The main theorem is: If U has continuous second derivatives for $|x|<\alpha$, and if the inequality

$$\int_{\Omega_R}|\Delta U|^2ds\leqq c\int_{\Omega_R}|U|^2ds,$$

where Ω_R is the p-dimensional sphere $|x|=R$, is satisfied uniformly in $|x|$ for $0<R\leqq\alpha$, then U vanishes identically if $\int_{\Omega_R}|U|^2ds=o(R^n)$ for all n, as $R\to 0$.

The theorem is proved in two steps. The rather complicated proof is based on a discussion of the functions $F_{n,j}(R)=\int_{\Omega_R} U(R\xi)S_{n,j}(\xi)ds$ where ξ denotes the unit vector in the direction of (x) and $S_{n,j}$ is a complete set of orthonormal spherical harmonics of order n in p dimensions. Obviously the theorem is valid for the solutions of the equation (1). For the solutions of the equation $\Delta U=F(x, U)$ the following theorem holds. Let U_1 and U_2 be two solutions having continuous second derivatives for $|x-x_0|\leqq\alpha$ and let $U_1(x_0)=U_2(x_0)$; furthermore, let $F(x, U)$ satisfy a Lipschitz condition in the neighborhood of (x_0) and $U_1(x_0)$. Then from

$$\int_{|x-x_0|=R}|U_1(x)-U_2(x)|^2ds=o(R^n)$$

for all n, as $R\to 0$ it follows that $U_1(x)=U_2(x)$ identically.

H. Bremekamp (Delft).

Bononcini, Vittorio E. Ancora sul problema di Dirichlet in domini rettangolari. Atti Sem. Mat. Fis. Univ. Modena 6 (1951–52), 16–33 (1953).

L'auteur poursuit ses raffinements [mêmes Atti 5, 154–164 (1951); ces Rev. 14, 750] des théorèmes de Cesari [Rend. Circ. Mat. Palermo 60, 185–212 (1936)] concernant la régularité des solutions du problème de Dirichlet relatif à l'équation $\Delta u-\lambda u=f$ et à un domaine rectangulaire $A\subset R^n$. Il montre que des hypothèses convenables sur f et la donnée-frontière entraînent que les dérivées premières et secondes de u sont lipschitziennes dans A tout entier (frontière comprise).

J. Deny (Strasbourg).

Birman, M. Š. On the theory of general boundary problems for elliptic differential equations. Doklady Akad. Nauk SSSR (N.S.) 92, 205–208 (1953). (Russian)

Some complements to the general theory of boundary problems due to Višik [Trudy Moskov. Mat. Obšč. 1, 187–246 (1952); these Rev. 14, 473] are given. The most important is the following. Consider an elliptic differential operator

$$L=-\sum\frac{\partial}{\partial x_i}a_{ik}(x)\frac{\partial}{\partial x_k}+c(x)$$

defined in an open bounded subset Ω of real n-space with a smooth boundary Γ and assume that its coefficients are sufficiently regular in $\Omega+\Gamma$ and that $c\geqq 0$. Let C^2 be the set of twice continuously differentiable functions in Ω and C_0^2 the set of all functions in C^2 having compact supports in Ω. Then if S and T are the closures of L when defined on C_0^2 and C^2 respectively, one has $T=S^*$. No proofs.

L. Gårding (Lund).

Birman, M. Š. On minimal functionals for elliptic differential equations of second order. Doklady Akad. Nauk SSSR (N.S.) 93, 953–956 (1953). (Russian).

In the notations of the preceding review, let $\tilde{S}$ be a positive definite self-adjoint extension of S. According to the theory of Višik (loc. cit.), the domain $D(\tilde{S})$ of $\tilde{S}$ may be characterized by $T_1(g)=T_2(g)=0$ where T_1 and T_2 are certain boundary operators and g belongs to $D(S^*)$. Quadratic functionals are constructed for which the minimizing element g satisfies various boundary-value problems, e.g., $\tilde{S}g=h$ (the class of admissible functions satisfying $S^*g=h$) or $S^*g=0$ with given $T_1(g)$ and $T_2(g)$ (the class of admissible functions satisfying $S^*g=0$). The classical boundary-value problems enter as special cases. No proofs. *L. Gårding.*

Šapiro, Z. Ya. On general boundary problems for equations of elliptic type. Izvestiya Akad. Nauk SSSR. Ser. Mat. 17, 539–562 (1953). (Russian).

Let L be an elliptic differential operator of the second order defined in an open subset D of real 3-space and let Λ be an arbitrary differential operator defined in a neighborhood of the boundary Γ of D. Assuming that L has constant and Λ smooth coefficients, the author reduces the problem of finding a solution u of $Lu=0$ in D with given Λu on Γ to a Fredholm equation with a suitable kernel. This kernel is

first found when D is a half-space and Γ is a plane and afterwards constructed for a general bounded D with a sufficiently smooth Γ. [Reviewer's remark. This method has been used also by Å. Pleijel, Proc. Oklahoma Symposium on Differential Problems, Okla. Agric. and Mech. Coll., Stillwater, Okla., 1951, pp. 413–437; these Rev. **13**, 948.] The results are generalized to certain equations and systems of higher order. They overlap with the more abstract and general results of Višik [Trudy Moskov. Mat. Obšč. **1**, 187–246 (1952); Doklady Akad. Nauk SSSR (N.S.) **82**, 181–184; **86**, 645–648 (1952); these Rev. **14**, 473, 279, 652].

L. Gårding (Lund).

Titchmarsh, E. C. Eigenfunction expansions associated with partial differential equations. IV. Quart. J. Math., Oxford Ser. (2) **4**, 254–266 (1953).

The present paper is concerned with showing how results established for the differential equation

$$\nabla^2\varphi+\{\lambda-q(x,y)\}\varphi=0$$

in the preceding papers of this series [Proc. London Math. Soc. (3) **1**, 1–27 (1951); **3**, 80–98, 153–169 (1953); these Rev. **13**, 241; **15**, 229] may be extended to an equation of the form

$$(*)\quad \frac{\partial}{\partial x}\left\{p(x,y)\frac{\partial\varphi}{\partial x}\right\}+\frac{\partial}{\partial y}\left\{p(x,y)\frac{\partial\varphi}{\partial y}\right\}+\{\lambda s(x,y)-q(x,y)\}\varphi=0.$$

The equation specifically considered is

$$(**)\qquad \nabla^2\psi+\{\lambda s_1(x,y)-q_1(x,y)\}\psi=0,$$

to which (*) is reducible by the substitution

$$\psi(x,y)=\{p(x,y)\}^{1/2}\varphi(x,y)$$

under the assumption that $p(x, y)$ is a positive continuous function of class C''. The Green's function for (**), with the region of consideration the entire (x, y) plane, is shown to be unique if $s_1(x,y)\geqq S(r)$, $q_1(x,y)\geqq -Q(r)$, with $r=(x^2+y^2)^{1/2}$, and $S(r)$, $Q(r)$ are functions of class C' such that

$$S(r)\geqq 0,\quad Q(r)\geqq\delta>0,\quad S'(r)=O\{S(r)\},$$
$$Q'(r)=O[\{Q(r)\}^{3/2}],\quad Q(r)/S(r)\to\infty$$

as $r\to\infty$, and $\int^\infty S(r)\{Q(r)\}^{-1/2}dr$ is divergent. The discreteness of the spectrum of (**) is proved under the assumption that $q_1(x,y)/s_1(x,y)\to\infty$ and $s_1(x,y)=O(r^k)$ for some $k>0$, as $r\to\infty$.

W. T. Reid (Evanston, Ill.).

✶Bicadze, A. V. K probleme uravnenii smešannogo tipa. [On the problem of equations of mixed type.] Trudy Mat. Inst. Steklov. vol. 41. Izdat. Akad. Nauk SSSR, Moscow, 1953. 59 pp. 3.15 rubles.

This monograph discusses various boundary-value problems for the equation

$$(*)\qquad \frac{\partial^2 U}{\partial x^2}+\operatorname{sgn} y\frac{\partial^2 U}{\partial y^2}=0.$$

Many of the results were announced earlier [Lavrent'ev and Bicadze, Doklady Akad. Nauk SSSR (N.S.) **70**, 373–376 (1950); Bicadze, ibid. **70**, 561–564 (1950); **78**, 621–624 (1951); Soobščeniya Akad. Nauk Gruzin. SSR **11**, 205–210 (1950); these Rev. **11**, 724; **14**, 280, 281].

For equation (*) an existence and uniqueness theorem is established for the Tricomi problem. That is, let σ be an arc in the upper half of the xy-plane with end-points at $(0, 0)$ and $(1, 0)$. Then σ and the lines L_1: $y=-x$, $0\leqq x\leqq\frac{1}{2}$, L_2: $y=1-x$, $\frac{1}{2}\leqq x\leqq 1$, enclose a domain D. The Tricomi problem concerns the existence and uniqueness of a solution of (*) in D which assumes prescribed values on σ and L_1.

Problem (T_1) is the following: Let σ be as described above and consider a sequence of points

$$A_k=(\tfrac{1}{2}a_k,\ -\tfrac{1}{2}a_k),\quad B_k=(\tfrac{1}{2}a_k+\tfrac{1}{2},\ \tfrac{1}{2}a_k-\tfrac{1}{2}),\quad k=0,1,\cdots,n+1,$$

where $0=a_0<a_1<\cdots<a_{n+1}=1$. Then problem T_1 is similar to the Tricomi problem in that values are prescribed on σ and on the segments A_kA_{k+1} for even k and B_kB_{k+1} for odd k. Problem T_2 extends the Tricomi problem to multiply-connected domains while problem T_3 treats the case where the normal derivative is prescribed on σ. Chapter 2 establishes existence and uniqueness theorems for problems T_1, T_2 and T_3.

For problem M the domain consists of the curve σ and line L_2 as above and a monotone arc Γ: $y=-\gamma(x)$ issuing from the origin and intersecting L_2. For this domain boundary values are prescribed on σ and Γ. In chapter 3 existence and uniqueness are established if σ, given parametrically by $x=x(s)$, $y=y(s)$, satisfies the additional restriction $y_s'(x-x^2-y^2)-yx_s'\geqq 0$.

M. H. Protter.

Protter, M. H. New boundary value problems for the wave equation and equations of mixed type. J. Rational Mech. Anal. **3**, 435–446 (1954).

L'auteur dans cet article formule de nouveaux énoncés de problèmes aux limites relatifs à des équations aux dérivées partielles linéaires à trois variables. Il a été conduit à ces énoncés en cherchant à formuler une généralisation du problème de Tricomi pour des équations simples du type mixte à trois variables indépendantes. Par exemple, un des problèmes envisagés par l'auteur est le suivant: trouver la solution de l'équation de ondes cylindriques $u_{xx}+u_{yy}=u_{zz}$ prenant des valeurs données sur le cercle $z=0$, $x^2+y^2\leqq z_0^2$, et sur la partie de la surface du cone $(z-z_0)^2=x^2+y^2$ définie par les inégalités $0\leqq z\leqq[(x-x_0)^2+(y-y_0)^2]^{1/2}$. La preuve des théorèmes d'unicité de la solution de ces nouveaux problèmes est obtenus par application de la puissante méthode que l'on commence à appeler "méthode A B C de Friedrichs" [Protter, même J. **2**, 107–114 (1953); ces Rev. **14**, 654]; cette même méthode permet également de démontrer le théorème d'unicité pour le problème généralisant le problème de Tricomi.

P. Germain (Paris).

Weinberger, Hans F. Sur les solutions fortes du problème de Tricomi. C. R. Acad. Sci. Paris **238**, 1961–1962 (1954).

Germain and Bader [O.N.E.R.A. Publ. no. **54** (1952); these Rev. **14**, 654] have shown the existence of a weak solution of the Tricomi problem for the Tricomi equation. This note establishes the fact that this weak solution is a strong solution.

M. H. Protter (Berkeley, Calif.).

Fourès-Bruhat, Yvonne. Résolution du problème de Cauchy pour des équations hyperboliques du second ordre non linéaires. Bull. Soc. Math. France **81**, 225–288 (1953).

In this paper, Cauchy's problem is solved for a hyperbolic system of quasilinear equations

$$(1)\qquad A^{\lambda\mu}\frac{\partial^2 u_s}{\partial x^\lambda\partial x^\mu}+f_s=0\quad (s=1,\cdots,N)$$

in N unknown functions u_s of n variables x^λ, the coefficients $A^{\lambda\mu}$ and f_s being assumed to be sufficiently differentiable functions of the x^λ and of the u_s and their first derivatives.

The initial data are assumed also to have a sufficient (finite) number of continuous derivatives. The author's procedure in the case of an even number of independent variables is analogous to that of her earlier paper [Acta Math. **88**, 141–225 (1952); these Rev. **14**, 756] on the case $n=4$. By Sobolev's method, a "formula of Kirchhoff" first is derived for the solution of Cauchy's problem for linear equations, the $A^{\lambda\mu}$ being functions of the independent variables only and the f_s linear with respect to the dependent variables and their first derivatives. This formula gives the value of the solution u_s at a point P in terms of integrals of the u_s over the backward characteristic conoid with vertex at P, and in terms of Cauchy data and their derivatives; it thus leads to a proof of the existence of the solution in the linear case as well as to estimates for the solution and its derivatives. Next, a system of quasi-linear equations of form (1) is considered, or rather, the system (1′) obtained from (1) by a certain number of differentiations. Formally as in the linear case, this system is reduced to integral equations which then are solved by a procedure of approximation in which the unknown functions in $A^{\lambda\mu}$ and f_s are replaced by known ones, convergence being established from the estimates previously derived. The fact that the solution of the integral equations satisfies the differential equations is shown by approximation by equations with analytic coefficients. Finally, the case of an odd number of independent variables is treated by the method of descent.
A. Douglis (New York, N. Y.).

***Fourès, Y. Résolution du problème de Cauchy pour des équations hyperboliques du second ordre non linéaires.** Premier colloque sur les équations aux dérivées partielles, Louvain, 1953, pp. 25–33. Georges Thone, Liège; Masson & Cie, Paris, 1954.

This paper is a summary of the paper reviewed above.
A. Douglis (New York, N. Y.).

Douglis, Avron. The problem of Cauchy for linear, hyperbolic equations of second order. Comm. Pure Appl. Math. 7, 271–295 (1954).

L'auteur expose une nouvelle méthode de résolution du problème de Cauchy pour les équations hyperboliques du second ordre à n variables d'espace ($n>2$), les coefficients, les données initiales et la surface qui porte ces dernières étant supposés suffisamment différentiables. Son but est d'étendre le procédé utilisé par H. Lewy dans le cas de l'équation des ondes à trois variables d'espace [Courant et Hilbert, Methoden der Mathematischen Physik, t. II, Springer, Berlin, 1937, pp. 370–371]. Le rôle principal est joué par une identité relativement compliquée entre deux fonctions suffisamment différentiables, faisant intervenir des dérivées internes au conoïde caractéristique rétrograde issu d'un point Q à l'instant t. En donnant à l'une des fonctions une expression convenable, l'auteur obtient une équation intégrale de Volterra, permettant de trouver la solution du problème de Cauchy en Q à l'instant t et ce, pour chaque valeur impaire de n. Il montre que cette dernière équation admet une solution unique et l'obtient par des itérations. La méthode ne paraît s'appliquer directement que pour les valeurs impaires de n et exige l'intervention de la méthode de descente pour n pair.
H. G. Garnir (Liège).

Friedrichs, K. O. Symmetric hyperbolic linear differential equations. Comm. Pure Appl. Math. 7, 345–392 (1954).

Dans certaines régions lenticulaires $\mathcal{R}$ de R^n, limitée par les cloisons $\mathcal{S}$ et $\mathcal{T}$, d'orientation d'espace par rapport à une des variables $x_1, \cdots, x_n$, l'auteur considère l'opérateur

$$E=\sum A_i(x_1, \cdots, x_n)(\partial/\partial x_i)+B,$$

$A_1, \cdots, A_n$ étant des matrices symétriques non négatives dont la somme est définie positive et B une matrice quelconque. Il généralise la notion de vecteur u, solution du problème de Cauchy, posé dans $\mathcal{R}$, pour l'équation $Eu=f$, la valeur de u étant donnée sur $\mathcal{S}$ et égale à g. Si $|u|$ et $|f|\in L^2(\mathcal{R})$, $|g|\in L^2(\mathcal{S})$, cette solution généralisée peut être définie en un sens faible ou en un sens fort. Au sens faible, la solution u est définie par la relation

$$\int_{\mathcal{R}}\left(\frac{\partial}{\partial x_i}A_i v, u\right)d\mathcal{R}=\int_{\mathcal{R}}(v, f)d\mathcal{R}+\int_{\mathcal{S}}(v, \sum n_i A_i g)d\mathcal{S},$$

quels que soient les vecteurs v convenables, identiquement nuls sur $\mathcal{T}$. D'autre part, u est dit solution forte s'il existe dans des vecteurs u_n continument différentiables dans $\mathcal{R}$ et sur $\mathcal{S}$ tels que $|u-u_n|$ et $|Eu_n-f|$ tendent vers 0 dans $L^2(\mathcal{R})$ et que $|u_n-g|$ tende vers 0 dans $L^2(\mathcal{S})$ quand n tend vers l'infini.

Grâce à une inégalité, dite de l'énergie,

$$\int_{\mathcal{R}}|u|^2d\mathcal{R}+\int_{\mathcal{T}}|u|^2d\mathcal{T}\leqq\gamma^2\int_{\mathcal{R}}|Eu|^2d\mathcal{R}+\gamma^2\int_{\mathcal{S}}|u|^2d\mathcal{S},$$

$$\gamma=\text{const.},$$

l'auteur établit l'unicité de la solution faible, il en déduit alors l'existence de la solution forte dont il démontre l'identité avec la solution faible.

La dernière partie du travail est consacrée à un théorème de différentiabilité. L'auteur montre que u admet des dérivées premières s'il en est ainsi de f et g à condition que les éléments des matrices A et B possèdent des dérivées continues.
H. G. Garnir (Liège).

John, Fritz. Solutions of second order hyperbolic differential equations with constant coefficients in a domain with a plane boundary. Comm. Pure Appl. Math. 7, 245–269 (1954).

Plaçons-nous dans l'espace euclidien à trois dimensions. Posons

$$L_{c,k}u=u_{tt}-c^2(u_{xx}+u_{yy}+u_{zz})+ku \quad (c>0).$$

Soit R un compact situé dans le plan $x=0$. Le couple des fonctions (f, g) définies dans R est dit compatible avec $L_{c,k}$ s'il existe dans un voisinage de R situé dans $x\geqq 0$ une fonction indéfiniment dérivable u telle que $u=f$, $u_x=g$ sur $x=0$. L'auteur montre que la condition nécessaire et suffisante pour qu'une solution $u\in C_\infty$ de $L_{c,k}$ puisse être prolongée respectivement par symétrie ou antisymétrie au travers du plan $x=0$ est que $(f, 0)$ ou $(0, g)$ soient compatibles avec $L_{c,k}$. Il obtient trois théorèmes indiquant quand cette circonstance se présente: (1) si $(f, 0)$ et $(0, g)$ sont compatibles avec $L_{c,k}$ dans R, $(f, 0)$ et $(0, g)$ le sont aussi avec $L_{c',k'}$ si $c'^2\leqq c^2$; (2) si (f, g) est compatible avec $L_{c,k}$, (af, bg) $(ab<0)$ avec $L_{c',k'}$, alors $(f, 0)$ et $(0, g)$ sont compatibles dans $R'\subset R$ avec $L_{c',k'}$, si $c'^2\leqq c^2$; (3) si (f, g) est compatible avec $L_{c,k}$ et $L_{c,k'}$, alors, dans $R'\subset R$, $(f, 0)$ et $(0, g)$ sont compatibles avec $L_{c,k'}$.

Dans les cas (2) et (3) où le couple (f, g) donne naissance à une solution prolongeable, l'auteur signale un procédé permettant de réaliser effectivement ce prolongement. Il applique son résultat à la résolution du problème de transmission au travers du plan $x=0$ consistant à chercher u et u', solutions respectives de $L_{c,k}u=0$ et $L_{c',k'}u'=0$ de part et

d'autre de $x=0$ avec les conditions de passage $u'=au$, $u_x'=bu$, connaissant les conditions initiales de u et de u' dans chaque demi-espace. De même, il résout le problème mixte de l'équation du quatrième ordre $L_{c,k}L_{c',k}u=0$, $c\neq c'$, avec les conditions aux limites $u=u_x=0$ sur $x=0$ et les conditions initiales $\partial^i u/\partial t^i=h_i(x, y, z)$, $i=0, 1, 2, 3$.

H. G. Garnir (Liège).

Hille, Einar. The abstract Cauchy problem and Cauchy's problem for parabolic differential equations. J. Analyse Math. 3, 81–196 (1954).

A comprehensive exposition is given of the Cauchy's problem for the parabolic equations (on the real line) from the view point of the semi-group theory. The contents read as follows. Chap. I. The abstract Cauchy problem. Chap. II. Basic theory of differential equations. Chap. III. The Cauchy problem for the diffusion equations. Chap. IV. Operators C having pure discrete point spectra. Chap. V. Restricted operators having pure discrete point spectra. Chap. VI. Extraneous solutions. Appendix.

Chap. I gives an abstract formulation of Cauchy's problem together with a criterion of uniqueness of the solutions. Chap. II. Discussion of the resolvents of the operators

$$C(u)=b(x)u''(x)+a(x)u'(x)$$

and

$$L(u)=((b(x)u(x))'-a(x)u(x))'$$

(with continuous coefficients $a(x)$ and $b(x)>0$) respectively in $C[\alpha, \beta]$ and $L(\alpha, \beta)$ where $-\infty\leqq\alpha<0<\beta\leqq\infty$. Chap. III. Theorem 7.5 states that C generates a semi-group in $C[\alpha, \beta]$ continuous at $t=0$ if and only if $\Omega(\alpha)=\Omega(\beta)=\infty$. Here

$$\Omega(x)=\int_0^x (W_1(x)-W_1(s))(b(s)W(s))^{-1}ds,$$

$$W_1(x)=\int_0^x W(s)ds,\quad W(x)=\exp\left(-\int_0^x a(s)b(s)^{-1}ds\right).$$

The condition corresponding to $L(u)$ is

$$\Omega(\alpha)=\Omega(\beta)=W_2(\alpha)=W_2(\beta)=\infty,$$

where $W_2(x)=\int_0^x W_1(s)(b(s)W(s))^{-1}ds$. These results were announced in a note of the author [C. R. Acad. Sci. Paris 230, 34–35 (1950); these Rev. 11, 256] without detailed proof. Chap. IV discusses the spectral properties of the operators C and L together with the expansion in the "theta series" of the transition density associated with the diffusion equations $\partial/\partial t=C$ and $\partial/\partial t=L$. The same problem is discussed in Chap. V when C (and L) is not the infinitesimal generator of the semi-group strongly continuous in the whole space $C[\alpha, \beta]$ (and $L(\alpha, \beta)$). Chap. VI is devoted to the discussion of the non-uniqueness and the "explosion at a finite $t>0$" of the solutions of the diffusion equations. The results reviewed so far are illustrated, in the Appendix, by two concrete equations: (i) Hermite equation: $a(x)=-2x$, $b(x)=1$, $\alpha=-\infty$, $\beta=+\infty$; (ii) Halm equation: $a(x)=0$, $b(x)=(x^2+1)^2$, $\alpha=-\infty$, $\beta=+\infty$.

K. Yosida (Osaka).

Dobryšman, E. M. On a particular case of a problem of heat conduction for two media. Akad. Nauk SSSR. Prikl. Mat. Meh. 18, 219–224 (1954). (Russian)

Let two media be separated by the plane $x=0$ and consider the problem of heat conduction in this case. Let T_1 and T_2 represent the temperature in the two media measured from arbitrary values $T_1^{(0)}$ and $T_2^{(0)}$. The differential equations are of the form

$$(1)\quad \begin{aligned}\frac{\partial T_1}{\partial t}&=\frac{\partial}{\partial x}\left(k_1(x)\frac{\partial T_1}{\partial x}\right)\quad (x\geqq 0),\\ \frac{\partial T_2}{\partial t}&=\frac{\partial}{\partial x}\left(k_2(x)\frac{\partial T_2}{\partial x}\right)\quad (x\leqq 0),\end{aligned}$$

where k_1 and k_2 are the coefficients of heat conduction in the two media. Let the initial temperatures be given by $T_1=T_2=0$ for $t=0$. The fact that there is no discontinuity in the temperature at the separating plane is expressed by the condition that $T_1=T_2$ for $x=0$ and the condition of thermal equilibrium by

$$-\lim_{x\to+0}\lambda_1(x)\frac{\partial T_1}{\partial x}+\lim_{x\to-0}\lambda_2(x)\frac{\partial T_2}{\partial x}+hT_1\bigg|_{x=0}=W(t).$$

Finally, it is required that

$$T_1\to 0\ (x\to\infty)\quad\text{and}\quad T_2\to 0\ (x\to-\infty).$$

After introduction of new independent variables, equations (1) take the form

$$(2)\quad \begin{aligned}2\left[\tau\frac{\partial T_1}{\partial\tau}-\zeta\frac{\partial T_1}{\partial\zeta}\right]\left[1+\sum_{n=1}^{\infty}\beta_n^{(1)}\tau^n\zeta^n\right]&=\frac{\partial^2 T_1}{\partial\zeta^2},\\ 2\left[\tau\frac{\partial T_2}{\partial\tau}-\xi\frac{\partial T_2}{\partial\xi}\right]\left[1+\sum_{n=1}^{\infty}\beta_n^{(2)}\tau^n\xi^n\right]&=\frac{\partial^2 T_2}{\partial\xi^2}.\end{aligned}$$

The author assumes a solution of equations (2) of the form

$$(3)\quad T_1(\tau, \zeta)=\sum_{n=0}^{\infty}T_{1n}(\zeta)\tau^n,\quad T_2(\tau, \xi)=\sum_{n=0}^{\infty}T_{2n}(\xi)\tau^n.$$

If the series defined by (3) are to be a solution of equations (2), then the coefficients T_{in} must each satisfy an ordinary second-order differential equation whose homogeneous part is of the form

$$(4)\quad y_n''+2xy_n'-2ny_n=0\quad (n=0, 1, 2, \cdots).$$

By differentiation, $y_n'=y_{n-1}$, $y_n''=y_{n-2}$, so that equation (4) gives a determination for y_n:

$$(5)\quad y_n=\frac{1}{2n}(y_{n-2}+2xy_{n-1}).$$

The functions y_0 and y_1 are then determined in terms of the complementary error function so that each y_n may be determined from (5). In each instance the complementary function must be combined with a particular solution T_{im}^* to give the solution T_{im}. This gives a step-by-step determination of the coefficients T_{im} in equations (3).

An example of heat transfer between land and air is given in which $k_1(x)$ and $\lambda_1(x)$ are linear functions of x and $k_2(x)$ and $\lambda_2(x)$ are constant.

C. G. Maple (Ames, Iowa).

Zeuli, T. Su alcuni problemi di propagazione del calore in una sfera. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. 87, 127–145 (1953).

Two heat-conduction problems for a sphere S of radius R are treated. Problem one involves finding a function $U=U(x, y, z, t)$ which interior to S satisfies the parabolic equation

$$(*)\quad \Delta_2 U-k^{-2}U_t=0,\quad t>0,$$

and at time $t=0$ takes on for $r^2=x^2+y^2+z^2<R^2$ the preassigned values $f(x, y, z)$. In the second problem U satisfies (*) for $r<R$ and on S assumes assigned values $F(x, y, z, t)$, in addition $U(x, y, z, 0)=0$ for $r<R$. Methods for solving

these problems can also be found in the book, Conduction of heat in solids [Oxford, 1947; these Rev. 9, 188] by H. S. Carslaw and J. C. Jaeger. (When the obvious misprint in equations (28') is corrected, the resulting equations, termed by the author functional equations, are Volterra integral equations of the second kind.) *F. C. Dressel.*

Manfredi, Bianca. Osservazioni su di un problema di distribuzione della temperatura in un mezzo che si muove. Rivista Mat. Univ. Parma 4, 327–335 (1953).

The author points out that the transformation

$$U(x, t) = T(x, t) \exp\left(-\frac{1}{2}\int_0^x v(\xi)d\xi\right)$$

changes the problem of finding a solution U of the heat-conduction problem for a fluid moving with velocity $v(x)$ in the half plane $x>0$:

(*) $U_{xx}+vU_x=K^{-1}U_t,\quad U(x, 0)=\varphi(x),\quad U(0, t)=\psi(t),$

into the problem of finding a solution T of the heat-conduction problem for a stationary fluid in a nonhomogeneous medium:

$$T_{xx}+f(x)T=K^{-1}T_t,\quad T(x, 0)=\varphi^*(x),\quad T(0, t)=\psi^*(t).$$

Here $f(x)=-\frac{1}{4}(2v_x+v^2)$. Solutions of (*) in the special cases $f(x)=0$ and $f(x)=-c^2$, with $\varphi(x)=Ax$, $\psi(t)=-\lambda t$, are written out in terms of a finite number of tabulated functions. *F. C. Dressel* (Durham, N. C.).

Mambriani, Antonio. Determinazione delle soluzioni razionali intere di particolari equazioni alle derivate parziali. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) 16(85), 127–132 (1952).

The author shows that every solution of the equation:

$$(xy+1)\frac{\partial^2 z}{\partial x\partial y}-nx\frac{\partial z}{\partial x}-my\frac{\partial z}{\partial y}+mnz=0\quad (m, n=0, 1, 2, \cdots)$$

which is an entire rational function of x, or of y, or of both variables, can be expressed in terms of particular rational functions of x (or y) whose coefficients are functions of y obtained by successive application of a certain operator: $y^n I_y y^{-n-1} D_y$ to an arbitrary function of y (or x). In certain cases, there is also some restriction on the degree of the solution. Here D_y is the derivative operator and I_y is its inverse operator so chosen that $D_y I_y = I_y D_y$.
D. L. Bernstein (Rochester, N. Y.).

Pini, Bruno. Traduzione in equazioni integrali di un problema analogo al problema biarmonico fondamentale. Rend. Sem. Mat. Univ. Padova 22, 192–206 (1953).

This paper deals with the problem of finding a solution $u(x, y)$ of:

$$(1)\qquad \frac{\partial^4 u}{\partial x^4}-2\frac{\partial^3 u}{\partial^2 x\partial y}+\frac{\partial^2 u}{\partial y^2}=0$$

which is regular in a region D bounded by a segment c of the characteristic line $y=0$, and two adjacent non-characteristic curves γ_i: $x=\chi_i(y)$ $(0\leqq y\leqq a)$ $(i=1, 2)$, and which satisfies the boundary conditions: (2) $u=g(x)$ and $u_y=\psi(x)$ on c; (3) $u=f_i(y)$ and $u_x=\varphi_i(y)$ and γ_i $(i=1, 2)$. The first theorem gives an explicit formula for a function u satisfying (1) and (2). The second theorem deals with solution of (1), (3) and (2'): $u=u_y=0$ on c; it is proved that there exist four uniquely determined functions such that a solution of (1), (2'), (3) is given by an explicit formula involving integrals of these four functions and the kernel:

$$U_{m,n}(x, y; \xi, \eta)=(x-\xi)^m(y-\eta)^{-n}\exp[-(x-\xi)^2(4y-4\eta)^{-1}].$$

The four functions are the solutions of a set of four integral equations, so the problem has been transferred from solving a differential equation to solving four integral equations.
D. L. Bernstein (Rochester, N. Y.).

Integral Equations

★Pogorzelski, Witold. Równania całkowe i ich zastosowania. Tom I. Własności ogólne równań Fredholma i Volterry. [Integral equations and their applications. Vol. I. General properties of the Fredholm and Volterra equations.] Pánstwowe Wydawnictwo Naukowe, Warszawa, 1953. 152 pp. 13.70 zł.

This is a textbook intended for graduate students in mathematics and physical sciences. The exposition is straightforward and clear and, as the subtitle indicates, is concerned only with general aspects of the theory. A second volume will treat special kernels and applications. The treatment of topics in this volume is similar to that in several standard treatises; no attempt is made to use tools of functional analysis for a possibly more perspicuous exposition. The chapter headings give an idea of the selection of topics: I) Volterra integral equations; II) Fredholm integral equation of the second kind; III) Weakly singular Fredholm equation; IV) Sets of orthogonal functions; V) Fredholm equation with symmetric kernel; VI) Fredholm equation of the first kind; VII) Special study of the resolvent kernel.
J. V. Wehausen (Providence, R. I.).

Germay, R. H. Sur une équation intégrale généralisant l'équation de première espèce de Volterra. Extension d'un théorème de Le Roux. Ann. Soc. Sci. Bruxelles. Sér. I. 68, 34–41 (1954).

The author proves the existence of a unique solution of the functional equation $u=\int_0^x\Phi[x, s, \varphi(s)]ds=f(x)$ under suitable hypotheses on the functional Φ. The method is based on a previous result [Mém. Soc. Roy. Sci. Liège (3) 13, no. 8 (1926); 14, no. 5 (1927)]. He also considers the equation $\Omega(x, u)=f(x)$, with u defined as above, and applies a procedure similar to that used by Le Roux [Ann. Sci. Ecole Norm. Sup. (3) 12, 227–316 (1895)] in solving a Volterra equation of the first kind. *I. A. Barnett.*

Germay, R. H. Sur des systèmes d'équations intégro-différentielles récurrentes de forme normale dont les termes intégraux contiennent les dérivées des fonctions inconnues. Ann. Soc. Sci. Bruxelles. Sér. I. 68, 5–12 (1954).

The author extends a result on a single integro-differential equation [same Ann. 67, 177–185 (1953); these Rev. 15, 630] to systems of such equations. *I. A. Barnett.*

Pliś, A. A uniqueness theorem for the solution of a family of hyperbolic integro-differential equations. Ann. Polon. Math. 1, 135–137 (1954).

The uniqueness theorem for a problem of Ważewski [Bull. Acad. Polon. Sci. Cl. III. 1, 79–82 (1953); these Rev. 15, 321] is reduced to that for the equation $u_{tt}-u_{xx}=0$.
F. A. Ficken (Knoxville, Tenn.).

Cherubino, Salvatore. Ancora sulle matrici infinite. Ann. Scuola Norm. Super. Pisa (3) 8, 77–80 (1954).

To meet criticisms contained in the remarks of the reviewer concerning a preceding paper [same Ann. (3) 6, 291–315 (1953); these Rev. 14, 1095] the author proves the following revision of a basic statement (p. 312 of paper): If A is the matrix $\{a_{ij}\}$, $i=1, \cdots, p$; $j=1, \cdots$; B the matrix $\{b_{jk}\}$, $j=1, \cdots$; $k=1, \cdots, p$ and if $c_{ik}=\sum_j a_{ij}b_{jk}$, convergence being assumed for $(i, k)=1, \cdots, p$, then

$$\det C=\sum_{(t)} \det A_{(t)} \det B^{(t)},$$

where $A_{(t)}$ is the matrix $\{a_{it_j}\}$ and $B^{(t)}$ the matrix $\{b_{t_jk}\}$, $(i, k)=1, \cdots, p$; $0<t_1<\cdots<t_p$. The "sum" $\sum_{(t)}$ is to be interpreted as $\lim_n \sum_{t_1t_2\cdots t_p}$ with $0<t_1<\cdots<t_p\leqq n$. Since this orders $\sum_{(t)}$ in a definite way, absolute convergence for the resulting series need not be assumed.

T. H. Hildebrandt (Ann Arbor, Mich.).

Functional Analysis

Nikodým, Otton Martin. On transfinite iterations of the weak linear closure of convex sets in linear spaces. Part B. An existence theorem in weak linear closure. (A study of convex sets in abstract linear spaces where no topology is supposed. IV.) Rend. Circ. Mat. Palermo (2) 3, 5–75 (1954).

[Part A appeared in same Rend. (2) 2, 85–105 (1953); these Rev. 15, 324.] In C. R. Acad. Sci. Paris 234, 1831–1833 (1952) [these Rev. 13, 753] the author stated and outlined a proof of the following theorem: If L is an infinite-dimensional linear space and α is an ordinal $<\Omega$, then L contains a convex set S for which $\operatorname{lin}^\alpha S\neq\operatorname{lin}^{\alpha+1} S$. The present paper gives the complete proof, with the aid of some geometrical notions in Hilbert space and a rather complicated transfinite induction.

V. L. Klee.

Dieudonné, Jean. On biorthogonal systems. Michigan Math. J. 2, 7–20 (1954).

As the author notes in the opening sentence, this paper is principally expository. Its value is derived from a systematic use of the modern (Bourbaki) approaches to topology and linear topological spaces. Again, as the author remarks, most results are straightforward generalizations of earlier work of Banach, James, Kaczmarz and Steinhaus, Karlin, Wilansky and the reviewer. The spaces under discussion are t spaces ("espace tonnelé"). The treatment centers around the degrees of regularity of (denumerable!) biorthonormal (BN) systems. These are defined as follows: Let $S=\{a_n, b_n\}$ be BN, $\{a_n\}\subset E$, $\{b_n\}\subset E^*$ (the dual of E). S is called quasi-regular if $|\sum_1^N x^*(a_n)b_n(x)|$ is bounded for each $\{x, x^*\}\in E\times E^*$. S is called weakly regular if $\sum_1^\infty x^*(a_n)b_n(x)$ converges for each $\{x, x^*\}$. S is called strongly regular in case $\sum_1^\infty b_n(x)a_n$ converges strongly to some $q(x)$ in E^{**}. Finally, S is called completely regular if $q(x)\in E$. Thus $\{a_n\}$ is a basis for E if it is completely regular and the (weak) linear closure A of $\{a_n\}$ is E. S is called perfect if it is quasi-regular and if the (weak*) linear closure B of $\{b_n\}$ is the strong linear closure B_1 of $\{b_n\}$. Theorem: If $\{a_n, b_n\}$ is perfect and if $\sum_1^\infty x^{**}(b_n)a_n$ converges to $q(x'')$ in E, all $x^{**}\in E^{**}$, then A and B are reflexive spaces and $A^*=B$. Many examples are provided to show that the various degrees of regularity and the various closures of different strengths are in fact distinguishable.

B. Gelbaum (Minneapolis, Minn.).

Alexiewicz, A., and Orlicz, W. Analytic operations in real Banach spaces. Studia Math. 14 (1953), 57–78 (1954).

The problem considered in this paper is that of developing a theory of real analytic functions for functions defined on one real Banach space to another. Since every real (B)-space X can be embedded in a complex (B)-space Z and since real homogeneous polynomials of degree n on X can be extended to be complex homogeneous polynomials of degree n on Z, it follows that a real analytic theory can be obtained by specializing the complex analytic theory. It can be argued that a real theory should have an independent development, and this is the point of view of the authors. They have used E. Hille's treatise on "Functional analysis and semi-groups" (Chap. IV) [Amer. Math. Soc. Colloq. Publ., v. 31, New York, 1948; these Rev. 9, 594] as a model. However, in place of the vector-valued analogue of a theorem due to Hartogs which was employed so effectively by Hille, here the authors make use of the vector-valued analogues of theorems by Leja on sequences of polynomials. One finds that the basic notions have lost some of their elegance in the real case. For instance, in defining what Hille calls a Gâteaux differential, the authors have been forced to assume that $[d^nF(x+th)/dt^n]_{t=0}=\delta^nF(x; h)$ is a homogeneous polynomial of order n whereas this is a consequence of the existence of the derivative in the complex case. The authors were reduced to proving real results from the corresponding complex results only in dealing with what Hille calls Fréchet analyticity. Finally it should be mentioned that the authors have been able in one or two instances to extend earlier results by exploiting properties of what they call "strictly fundamental sets of functionals".

R. S. Phillips.

Kurzweil, J. A characterization of analytic operations in real Banach spaces. Studia Math. 14 (1953), 82–83 (1954).

Using some results of the paper reviewed above, and an inequality due to S. N. Bernstein, the author proves the following generalization of a theorem of Bernstein: Let X and Y be real Banach spaces, G an open set in X, and F a function on G to Y. Then F is analytic in G if and only if for each $x_0\in G$ there is a neighborhood of x_0, two numbers $C>0$, $1>q>0$ (depending on x_0) and a sequence $\{P_n\}$ of polynomials such that P_n is of degree n and the inequality $\|F(x)-P_n(x)\|<Cq^n$ holds for every x in the neighborhood and $n=1, 2, \cdots$.

R. G. Bartle (New Haven, Conn.).

Cronin, Jane. Branch points of solutions of equations in Banach space. II. Trans. Amer. Math. Soc. 76, 207–222 (1954).

L'autore considera in uno spazio di Banach l'equazione (1) $x+(C+T)x=y$, dove C è un operatore lineare completamente continuo e T è un operatore nullo nell'origine e tale che in un intorno di questa sia:

$$\|T(u)-T(v)\|=O[(\|u\|+\|v\|)\|u-v\|].$$

Per $y=0$ la (1) ammette la soluzione $x=0$ e se questa è isolata è possibile definire per essa una moltiplicità m. Se m è diversa da zero la (1) ammette almeno una soluzione x per ogni y abbastanza prossimo a zero e addirittura m soluzioni per infiniti valori di y. Questi risultati, insieme ad altri che sarebbe troppo lungo riportare, fanno seguito a precedenti ricerche dell'autore [gli stessi Trans. 68, 105–131; 69, 208–231 (1950); questi Rev. 11, 361; 12, 716] che vengono qui semplificate e precisate. Sono anche indicate alcune applicazioni alla teoria delle equazioni integrali non lineari e a

quella delle equazioni ellittiche. Per queste ultime si ottiene una generalizzazione di un teorema di J. Schauder [Math. Ann. 106, 661–721 (1932)]. *C. Miranda* (Napoli).

Michal, A. D., and Hyers, D. H. Solutions of differential equations as analytic functionals of the coefficient functions. Acta Math. 91, 75–86 (1954).

Let B denote a complex Banach space and I the real interval $|\tau-\tau_0|\leqq\alpha$. Let Y be the set of all continuous functions on I to B, and let X be the set of bounded continuous functions $x(\tau, y)$ on $I\times\{y \in B: \|y-y_0\|\leqq\beta\}$ to B such that for each τ, $x(\tau, y)$ is an analytic function on $\|y-y_0\|<\beta$ to B. With the supremum as norm these two spaces are complex Banach spaces. It is readily seen that if $x_0 \in X$ and if $\|x\|<\beta\alpha^{-1}$, then the equation $dy/d\tau=x_0(\tau, y)$, $y(\tau_0)=y_0$, has a unique solution in Y. The authors give conditions under which the solution $y \in Y$ of $dy/d\tau=x(\tau, y)$, $y(\tau_0)=y_0$, depends analytically on $x \in X$ in a neighborhood of x_0. This theorem is proved by using the general implicit function theorem of T. H. Hildebrandt and L. M. Graves [Trans. Amer. Math. Soc. 29, 127–153 (1927)] which is proved here for analytic functions of Banach-space variables by "power series" methods. *R. G. Bartle* (New Haven, Conn.).

Berman, D. L. On certain linear operators carrying periodic functions into trigonometric polynomials. Doklady Akad. Nauk SSSR (N.S.) 95, 213–216 (1954). (Russian)

More results are announced, proofs being sketched, of special properties of the operators specified in the title. [See also earlier notes of the author, same Doklady (N.S.) 85, 13–16 (1952); 88, 9–12 (1953); these Rev. 14, 57, 767.] *E. Hewitt* (Seattle, Wash.).

Ladyženskiĭ, L. A. General conditions of complete continuity of P. S. Uryson's operator acting in a space of continuous functions. Doklady Akad. Nauk SSSR (N.S.) 96, 1105–1108 (1954). (Russian)

In continuation of a series of earlier papers by other writers, the author derives conditions under which the nonlinear integral operator defined by $A\varphi=\int_G K(x, y, \varphi(y))dy$ (G a compact set in n-space) is completely continuous in the space C of continuous functions $\varphi(y)$. The main result is: Let $K(x, y, u)$ $(x, y \in G, -\infty<u<\infty)$ satisfy: (a) K is continuous in u for all x and for almost all y in G and measurable in y for all x in G, $-\infty<u<\infty$; (b) $\int_G \sup_{|u|\leqq a} |K(x, y, u)|dy<\infty$ for all $a>0$; (c) for all $a>0$ and for all $x \in G$,

$$\lim_{\substack{|h|\to 0\\ x+h \in G}} \int_G \sup_{|u|\leqq a} |K(x+h, y, u)-K(x, y, u)|dy=0.$$

Then A is completely continuous. Analogous results are derived for complete continuity in the space C when A is of the form: $A\varphi=\int_G K(x, y)f(y, \varphi(y))dy$. *B. Gelbaum.*

Halmos, Paul R., and Lumer, Günter. Square roots of operators. II. Proc. Amer. Math. Soc. 5, 589–595 (1954).

[For part I see Halmos, Lumer and Schäffer, same Proc. 4, 142–149 (1953); these Rev. 14, 767.] If H is an infinite-dimensional complex Hilbert space, there is an invertible operator a with no square root and such that some neighborhood of a in the bound metric consists of invertible operators with no square root. *S. Sherman* (Philadelphia, Pa.).

Štraus, A. V. Generalized resolvents of symmetric operators. Izvestiya Akad. Nauk SSSR. Ser. Mat. 18, 51–86 (1954). (Russian)

This paper presents an extension and generalization, with much simpler and more detailed proofs, of the results described in two previous short notes of the author [Doklady Akad. Nauk SSSR (N.S.) 78, 217–220 (1951); 82, 209–212 (1952); these Rev. 12, 837; 13, 755]. The generalization consists in not requiring the operator to be defined on a dense subset of Hilbert space H. The principal theorem characterizes the generalized resolvents of a closed symmetric operator on H as follows: For each complex λ with $\mathrm{Im}\,(\lambda)\neq 0$ let there be given a linear operator R_λ defined throughout H. Then in order that the family $\{R_\lambda\}$ be a generalized resolvent of some closed symmetric operator with deficiency index m in the half-plane π it is necessary and sufficient that: (1) for every $\lambda_0 \in \pi$ there exists a subspace $L\subset H$ with deficiency index m such that

(a) $$R_\mu f-R_{\lambda_0}f=(\mu-\lambda_0)R_\mu R_{\lambda_0}f$$

for every $\mu(\mathrm{Im}\,(\mu)\neq 0)$ and every $f \in L$,

(b) $$\|R_\lambda\Psi\|^2\leqq\tau^{-1}\,\mathrm{Im}\,(R_\lambda\Psi, \Psi)$$

for every $\lambda \in \pi$ and every $\Psi \in H\ominus L$ $(\tau=\mathrm{Im}\,(\lambda))$, (c) $R_\lambda\Psi$ is a regular vector-valued function of λ in π for every fixed $\Psi \in H\ominus L$, (d) there exists a sequence $\lambda_n=\sigma_n+i\tau_n$ with $\lim \lambda_n=\infty$ and $\sup|\sigma_n/\tau_n|<+\infty$ and such that for every $\varphi \in H\ominus R_{\lambda_0}L$ we have

$$\lim \lambda_n(R_{\lambda_n}\varphi, \varphi)=-(\varphi, \varphi);$$

(2) for any complex number λ in the half-plane π, $R_\lambda{}^*=R_{\bar\lambda}$. *B. Crabtree* (Durham, N. H.).

Livšic, M. S. On spectral decomposition of linear nonselfadjoint operators. Mat. Sbornik N.S. 34(76), 145–199 (1954). (Russian)

This paper contains full proofs of the results announced by the author in two earlier notes [Doklady Akad. Nauk SSSR (N.S.) 84, 873–876, 1131–1134 (1952); these Rev. 14, 184, 185]. These results constitute an important advance in the spectral theory of non-Hermitian (and indeed non-normal) linear operators in Hilbert space.

In addition to the results summarized in the reviews of the preliminary notes, the following may be mentioned (notation and terminology as in earlier reviews).

Let

$$W(\lambda)=I+2i\,\mathrm{sgn}\,\mathfrak{J}(A)|\mathfrak{J}(A)|^{\frac12}(A^*-\lambda I)^{-1}|\mathfrak{J}(A)|^{\frac12},$$

the functions of the operator $\mathfrak{J}(A)$ being defined in the usual way by using its eigenvalues. Let (e_j) be an orthonormal base of the closed subspace E, consisting of eigenvectors of $\mathfrak{J}(A)$, and let the matrices of $W(\lambda)$, $2\mathfrak{J}(A)$, $[2|\mathfrak{J}(A)|]^{\frac12}$ and $\mathrm{sgn}\,\mathfrak{J}(A)$ with respect to this base be $w(\lambda)$, ω, $|\omega|^{\frac12}$ and J respectively. Then

$$w(\lambda)=I+i|\omega|^{\frac12}[((A^*-\lambda I)^{-1}e_k, e_j)]|\omega|^{\frac12}J$$

is the characteristic matrix function of A. If the subspace G_d reduces to $\{0\}$, the operator A is said to be simple; a simple operator is characterized, up to unitary equivalence, by its characteristic matrix function $w(\lambda)$.

The characteristic matrix function of a triangular model **A** is equal to the value $W(l, \lambda)$ at l of the generalized Wronskian matrix $\{W(k, \lambda), W(x, \lambda)\}$ of the system of difference

and differential equations

$$y(k+1)-y(k)=iy(k)\beta(k)[\alpha(k)I-\tfrac{1}{2}i\beta(k)J\beta(k)-\lambda I]^{-1}\beta(k)J \quad (k=1, 2, \cdots),$$

$$\frac{dy}{dx}=\frac{iy(x)\beta^2(x)J}{\alpha(x)-\lambda} \quad (0\leqq x\leqq l),$$

W being defined as the matrix solution of the above system satisfying the conditions

$$W(1, \lambda)=I, \quad W(\infty, \lambda)=W(0, \lambda).$$

An explicit solution of this system is obtained in terms of an infinite product and a 'multiplicative Stieltjes integral'. Necessary and sufficient conditions are obtained for an $r\times r$ matrix function $w(\lambda)$ to be the characteristic matrix function of an operator A of class $(i\Omega)$ and finite non-Hermitian rank r.

The proof of the main theorem uses approximation by operators defined in finite-dimensional spaces, the approximation being in terms of the convergence of the corresponding matrix functions, uniformly except near the spectrum of A^*.

A number of interesting applications of the main theorem are given. If A is an integral operator of the form

$$A\varphi=\int_a^b K(x, s)\varphi(s)ds,$$

K being a bounded kernel, and $\lambda=0$ is the only singular point of $(A-\lambda I)^{-1}$, then A is unitarily equivalent to a generalized Volterra operator:

$$g=\mathrm{A}f=i\int_0^l f(t)\beta(t)J\beta(x)dt,$$

where f and g are vectors in $\mathfrak{H}_r$, and $\beta(t)$ is an $r\times r$ matrix for each value of t.

If an operator A of the class $(i\Omega)$ has non-Hermitian signature $(p, 0)$, then

$$\sum_{j=1}^{\infty}\mathfrak{J}(\lambda_j)\leqq \mathrm{Tr}\,\mathfrak{J}(A),$$

equality holding if and only if the closed subspace generated by the finite-dimensional invariant subspaces of A is the whole space $\mathfrak{H}$. This result is applied to completeness problems for the eigenfunctions of homogeneous linear differential equations.

A number of special types of operators are analysed in detail; among these are the Jacobi matrices associated with Chebyshev polynomials. *F. Smithies.*

Gårding, L., and Wightman, A. Representations of the anticommutation relations. Proc. Nat. Acad. Sci. U. S. A. 40, 617–621 (1954). See the concerning note on p. 1336.

The authors give without proof a representation for all sets $\{a_n\}$ of bounded operators on a separable Hilbert space satisfying the relations $a_ja_k+a_ka_j=0$, $a_ja_k^*+a_k^*a_j=\delta_{jk}$, $j, k=1, 2, \cdots$. Their formulation is in terms of an occupation-number representation involving an operator-valued 'multiplier' and a measure on a space of possible occupation statistics. In the corresponding finite-dimensional question it is well known that the multiplier can be made trivial by a unitary transformation, but the extent to which this is the case when the dimension is infinite is not treated.

I. E. Segal (Chicago, Ill.).

Gårding, L., and Wightman, A. Representations of the commutation relations. Proc. Nat. Acad. Sci. U. S. A. 40, 622–626 (1954). See the concerning note on p. 1336.

The authors give without proof a representation for all sets of unitary operators $U_n(a)$ and $V_n(a)$ on a separable Hilbert space satisfying the relations: $U(0)=V(0)=I$, $U_n(a+b)=U_n(a)U_n(b)$, $V_n(a+b)=V_n(a)V_n(b)$, all $U_n(a)$ commuting $(n=1, 2, \cdots; -\infty<a<\infty)$, all $V_n(a)$ commuting, the $U_n(a)$ and the $V_m(b)$ commuting whenever $n\neq m$, and $U_n(a)V_n(b)=\exp[iab]V_n(b)U_n(a)$. As in their treatment of the anticommutation relations, the formulation is in terms of an occupation number representation involving an operator-valued 'multiplier' and a measure on a space of occupation statistics. The question of whether the multiplier can be trivialized by a unitary transformation is not examined, though this is known to be the case for the corresponding finite-dimensional question. *I. E. Segal.*

Deprit, André. Algèbre symétrique et seconde quantification d'un système de bosons. Ann. Soc. Sci. Bruxelles. Sér. I. 68, 23–33 (1954).

The author defines the algebra of symmetric tensors over a Hilbert space and obtains elementary properties relating to the second quantization for bosons. The work is parallel to that of Valatin on fermions [J. Phys. Radium (8) 12, 131–141 (1951); these Rev. 12, 784] and is largely contained in work of Cook [Trans. Amer. Math. Soc. 74, 222–245 (1953); these Rev. 14, 825] of which the author is apparently unaware. *I. E. Segal* (Chicago, Ill.).

Umegaki, Hisaharu. Decomposition theorems of operator algebra and their applications. Jap. J. Math. 22 (1952), 27–50 (1953).

This is a detailed account of work previously summarized [Proc. Japan Acad. 27, 328–333, 501–505 (1951); 28, 29–31 (1952); these Rev. 13, 756; 14, 58, 59] and also includes some new material. The novelty of the paper lies chiefly in its systematic examination of 'D^*-algebras', which are complex normed *-algebras with an approximate identity. The relations and decompositions of their traces, semi-traces, motions, and one- and two-sided representations are treated, with results closely parallel to the known ones for other types of algebras. *I. E. Segal.*

Kaplansky, Irving. Ring isomorphisms of Banach algebras. Canadian J. Math. 6, 374–381 (1954).

This paper is devoted to the proof of the following theorem: If ϕ is a ring isomorphism from one semi-simple Banach algebra A onto another, then A is a direct sum $A_1+A_2+A_3$ with A_1 finite-dimensional, ϕ linear on A_2, and ϕ conjugate linear on A_3. This generalizes results of Arnold [Ann. of Math. (2) 45, 24–49 (1944); these Rev. 5, 147] and Rickart [Duke Math. J. 18, 27–39 (1951); these Rev. 14, 385]. The proof is based on lemmas to the effect that each infinite-dimensional semi-simple Banach algebra contains an element with infinite spectrum and that in a primitive Banach algebra each additive endomorphism commuting with all left and right multiplications can be represented as multiplication by a member of the base field of the algebra. *E. L. Griffin, Jr.* (Ann Arbor, Mich.).

Nakano, Hidegorô. Product spaces of semi-ordered linear spaces. J. Fac. Sci. Hokkaido Univ. Ser. I. 12, 163–210 (1953).

The author proves many theorems concerning the set of bilinear functionals $\phi(x, y)$ where x and y vary over vector

lattices R and S, respectively. Using these bilinear functionals, he defines and studies products (vector lattices) which can be formed from the factors R, S, including the upper product RS and the lower product $R\times S$. Let $\widetilde{R, S}$ denote the conditionally complete vector lattice of those ϕ which can be represented as the difference of two non-negative bilinear functionals. A special role is played by those ϕ which have the form $\bar{x}\bar{y}$, i.e. $\phi(x, y)=\bar{x}(x)\bar{y}(y)$ for some $\bar{x}$, $\bar{y}$ in $\bar{R}$ and $\bar{S}$ respectively. A ϕ is called singular if it is orthogonal to all such $\bar{x}\bar{y}$. An important theorem is then proved characterizing those ϕ which are orthogonal to all singular ϕ; the proof of this theorem uses the author's integration theory for vector lattices. Theorems are also proved for the product spaces when the factors are (i) normed or (ii) modulared. *I. Halperin* (Utrecht).

Yamamuro, Sadayuki. Exponents of modulared semi-ordered linear spaces. J. Fac. Sci. Hokkaido Univ. Ser. I. **12**, 211–253 (1953).

H. Nakano, in one chapter of his "Modulared semi-ordered linear spaces" [Maruzen, Tokyo, 1950; these Rev. **12**, 420], gave a discussion of modular exponents for modulars of unique spectra. The present author gives a detailed theory for the general case, relating certain properties of the exponents to uniform convexity and uniform evenness of the lattice and generalizing in a nontrivial way, many theorems due originally to Nakano.

The following definitions are used: $m(a)$ is the modular function; $\pi(\xi/a)=\lim\,[m((\xi+\epsilon)a)-m(\xi a)]/\epsilon$ as $\epsilon\to 0$ if $m(\xi a)$ is finite, $=\infty$ otherwise; $\varphi_a(\alpha, \xi)=\pi(1/\xi a)/\xi^\alpha$ for $\alpha\geqq 1$, $\xi>0$; $\chi^\pi(a)=\inf\alpha$ with $\alpha\geqq 1$ and $\varphi_a(\alpha, \xi)$ decreasing in ξ; $\chi_\pi(a)=\sup\alpha$ with $\alpha\geqq 1$ and $\varphi_a(\alpha, \xi)$ increasing in ξ; $\chi^m(a)=\inf\alpha$ with $\alpha\geqq 1$ and $m(\xi a)/\xi^\alpha$ decreasing in ξ; $\chi_m(a)=\sup\alpha$ with $\alpha\geqq 1$ and $m(\xi a)/\xi^\alpha$ increasing in ξ.

Among the theorems proved:

$$\chi_\pi(a)m(\xi a)/\xi\leqq\pi(\xi/a)<\chi^\pi(a)m(\xi a)/\xi$$

and if $m(\xi a)=\xi^p m(a)$ for all $\xi>0$ then $\chi^\pi(a)=\chi_\pi(a)=p$. Conversely, if a is simple and $m(\xi a)$ is differentiable in $\xi>0$, then $\pi(\xi/a)=pm(\xi a)/\xi$ for all $\xi>0$ implies $m(\xi a)=\xi^p m(a)$ for all ξ. Also [reference is made to Burkill, Proc. London Math. Soc. (2) **28**, 493–500 (1928); Cooper, ibid. **26**, 415–432 (1927); and Mulholland, ibid. **33**, 481–516 (1932)] a is upper bounded if and only if $\chi^m(a)$ is finite.

Exponents are defined for subsets of the lattice, e.g. $\chi^m(S)=\sup\chi^m(a)$ for all a in S; $\chi_m(S)=\inf\chi_m(a)$ for all a in S. Omitting (S) when S is the entire lattice, the following relation holds for the conjugate modular:

$$\frac{1}{\chi^m}+\frac{1}{\chi_{\bar{m}}}=\frac{1}{\chi_m}+\frac{1}{\chi^{\bar{m}}}=1.$$

Generalizations are given for theorems proved by Orlicz for the spaces denoted by him as L^M spaces. *I. Halperin.*

Bauer, Heinz. Eine Rieszsche Bandzerlegung im Raum der Bewertungen eines Verbandes. S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. **1953**, 89–117 (1954).

The author generalizes the notion of valuation of a lattice [G. Birkhoff, Lattice theory, Amer. Math. Soc. Colloq. Publ., vol. 25, rev. ed., Amer. Math. Soc., New York, 1948, p. 74; these Rev. **10**, 673] and obtains decompositions of such valuations by use of F. Riesz's bands in complete vector lattices [see for example N. Bourbaki, Eléments de mathématique, XIII, Première partie, Livre VI, Ch. I–IV, Actualités Sci. Ind., no. 1175. Hermann, Paris, 1952, pp. 17–40; these Rev. **14**, 960]. Let V be a lattice, with sup and inf denoted by $\vee$ and $\wedge$ respectively. Let W be a complete vector lattice over an ordered field K (one assumes the compatibility of the order and operations in W with those of K). A function f with domain V and range contained in W is called a valuation if $f(a\vee b)+f(a\wedge b)=f(a)+f(b)$ for all a, $b\in V$. For a given $c\in V$, the set Ψ_c of all valuations f with $f(c)=0$ forms a vector lattice under the natural definitions of addition and scalar multiplication and with $f\leqq g$ if $f-g$ is isotone. The concept of relatively bounded variation is defined in the usual way [see Birkhoff, loc. cit., p. 83] and it is shown that the set Φ_c of all elements of Ψ_c having relatively bounded variation is a complete vector lattice. Continuity of a valuation is defined in terms of directed systems of elements attached to each element s of V, in a natural way. The set $\Phi_c{}^s$ of all continuous $f\in\Phi_c$ is shown to be a band in Φ_c. Riesz's theorem on bands and their complements [see Bourbaki, loc. cit., pp. 25–26] is applied to show that every valuation $f\in\Phi_c$ can be uniquely written as a sum f_s+f_u, where f_s is continuous in the sense described and f_u is purely discontinuous in the sense that $|g|\leqq|f_u|$ and g continuous imply $g=0$. 2-valued real valuations receive special attention, and a number of concrete examples are worked out. Among these is the case of a finitely additive measure on an algebra of sets, for which the notion of pure discontinuity was introduced and the present theorem proved by K. Yosida and the reviewer [Trans. Amer. Math. Soc. **72**, 46–66 (1952); these Rev. **13**, 543].

E. Hewitt (Seattle, Wash.).

Calculus of Variations

✶Dedecker, Paul. Calcul des variations, formes différentielles et champs géodésiques. Géométrie différentielle. Colloques Internationaux du Centre National de la Recherche Scientifique, Strasbourg, 1953, pp. 17–34. Centre National de la Recherche Scientifique, Paris, 1953.

The author describes his theory of multidimensional calculus of variations problems, with the use of exterior differential forms, fiber bundles and sheaves [cf. also Colloque de topologie et géométrie différentielle, Strasbourg, 1952, no. 3, Bibliothèque Nat. et Univ. Strasbourg, 1953; these Rev. **15**, 804]. A free calculus of variations problem in a manifold V_n is given by a p-form ω; the problem is to find the extremal chains (made up of singular differentiable cubes), i.e., those for which the first variation, with the boundary held fixed, vanishes. The extremal varieties are characterized as the integrals of the so-called first associated system of ω. A bound problem consists of a form ω and an ideal $\mathfrak{A}$ in the algebra of differential forms (or an ideal in the sheaf of germs of differential forms); the variations of chains are then required to be integrals of $\mathfrak{A}$ for each value of the variation parameter. The main construction of the theory is that of a space $\mathfrak{E}$, which is a fiber space, with a Euclidean space as fiber, over the space $V_n{}^p$ of p-contact elements in V_n as base space, and a form in $\mathfrak{E}$ whose extremal chains project into the extremals of the given bound problem in V_n. The coordinates in the fiber form are essentially Lagrange multipliers. The procedure consists roughly in considering some partial derivatives as new variables, and then adding, with Lagrange multipliers, side conditions, which express that these new variables are actually partial derivatives (in the usual one-dimensional case this amounts to taking y' as new variable and $dy-y'dx$ as a side condi-

tion). There is also a connection with E. Cartan's invariant integral. In the course of the argument there appear generalizations of all the classical concepts, mostly in the language of fiber bundles: Geodesic field, excess function, Lagrangian, canonical coordinates, Hamiltonian, non-regular elements, energy equation, Hamilton-Jacobi equations (the latter are partial differential equations with a $(p-1)$-form as unknown). It is indicated briefly how the existing theories fit into the present development. *H. Samelson.*

Krull, Wolfgang. Zur Variationsrechnung. Arch. Math. 5, 81–91 (1954).

Consider an integral of the form

$$I=\int_T L(t, x(t), p(t))dt,$$

where $x(t)$ denotes n functions $x_1(t), \cdots, x_n(t)$ of m variables $t=(t_1, \cdots, t_m)$ and $p(t)$ denotes the first partial derivatives $p_{i\alpha}=\partial x_i/\partial t_\alpha$. By a local minimizing surface is meant one to each point of which there is a neighborhood such that the surface affords a weak relative minimum to I relative to variations that vanish outside this neighborhood. The author shows in an interesting manner that a surface of class C'' is a local minimizing surface if (1) it satisfies the Euler equations and (2) if the inequality $L_{p_{i\alpha}p_{j\beta}}\pi_{i\alpha}\pi_{j\beta}\geqq h\pi_{i\alpha}\pi_{i\alpha}$ holds along the minimizing surface for all $(\pi_{i\alpha})$ and for some positive number h. He points out that (2) with $h=0$ is not a necessary condition when $m\geqq 2$ and $n\geqq 2$, as has been previously observed. The results here described are a consequence of results obtained by a different method by Van Hove [Nederl. Akad. Wetensch., Proc. 50, 18–23 (1947); Acad. Roy. Belgique. Cl. Sci. Mém. Coll. in 8° (2) 24, no. 5 (1949), p. 14; these Rev. 8, 522; 11, 730]. Van Hove has shown that the above result is valid when (2) is weakened by imposing the restriction that the matrices $(\pi_{i\alpha})$ appearing in the inequality be of rank one. In this form the corresponding condition with $h=0$ is also a necessary condition.
M. R. Hestenes (Los Angeles, Calif.).

Kruskal, Martin. The bridge theorem for minimal surfaces. Comm. Pure Appl. Math. 7, 297–316 (1954).

Largely using methods presented by R. Courant [Dirichlet's principle . . . , Interscience, New York, 1950; these Rev. 12, 90], the author establishes a result concerning minimal surfaces, which may be stated loosely as follows: If two given contours bound two given simply connected surfaces of relatively minimum area, then a new contour, formed by omitting a small arc on each given contour and connecting the two newly created endpoints of the one with those of the other by two close nearly parallel arcs, bounds a simply connected surface of relatively minimum area consisting of three parts, two of them close to the two given surfaces and the third of small area.
E. F. Beckenbach (Los Angeles, Calif.).

Emerson, R. C. On maximizing an integral with a side condition. Proc. Amer. Math. Soc. 5, 291–295 (1954).

The problem considered is to maximize

$$\mathfrak{F}(p)=\int F[p(x), x]dx$$

subject to $\mathfrak{G}(p)=\int G[p(x), x]dx=c$, where x ranges over a compact subset of the reals and $F(p, x)$, $G(p, x)$ are continuous. The class $\mathcal{P}$ of admissible functions comprises all real-valued $p(x)$ satisfying $u(x)\leqq p(x)\leqq v(x)$, where $u(x)$ and $v(x)$ are a fixed pair of continuous functions, for which the integrands in question are measurable. The main result is that if $\inf_p \mathfrak{G}(p)<c<\sup_p \mathfrak{G}(p)$, then a solution $p_0(x)$ and a θ in the open interval $(-\pi/2, \pi/2)$ exist such that, for each $x, p_0(x)$ maximizes $\cos\theta F(p, x)+\sin\theta G(p, x)$ among all p such that $u(x)\leqq p\leqq v(x)$. Using different methods the reviewer has obtained analogous results ~~[in an unpublished paper]~~ of considerably greater generality, especially in that any finite number of side conditions of the form $\mathfrak{G}_i(p)=c_i$ are allowed. *W. H. Fleming* (Santa Monica, Calif.).
[Ann. Math. 60, 578–594 (1954); MR 16, 732].

Saaty, Thomas, and Gass, Saul. Parametric objective function. I. J. Operations Res. Soc. Amer. 2, 316–319 (1954).

This paper discusses the problem of determining the dependence of the solution of a minimization problem involving linear functions and linear constraints upon parameters in the functions and constraints. *R. Bellman.*

Theory of Probability

Basu, D., and Laha, R. G. On some characterizations of the normal distribution. Sankhyā 13, 359–362; addenda 14, 180 (1954).

Let $x_1, x_2, \cdots, x_n$ be a sample from a normal population. It is then known that any translation-invariant statistic $g(x_1, x_2, \cdots, x_n)$ is independent of $L=\sum_{\alpha=1}^n x_i$. The authors state that the converse is in general not true. To show this they prove that the statistic $g(x_1, x_2)=0$ if $x_1<x_2$, $=1$ if $x_1\geqq x_2$ is independent of x_1+x_2 for any population with absolutely continuous distribution function. This, however, is not sufficient to prove their statement since it is conceivable that such an example could not be constructed for $n>2$. In the second part of the paper, they prove that the independence of k_r and k_1 implies the normality of the population provided that the moments of the population up to order r exist. Here k_r is the k-statistic introduced by R. A. Fisher [Proc. London Math. Soc. (2) 30, 199–238 (1929)]. In the addenda the authors fill a minor gap in their proof and give some generalizations. They indicate for instance that the assumption of full independence of k_r and k_1 could be replaced by the assumption that the conditional expectation of k_r, given k_1, equals the r-th cumulant. *E. Lukacs.*

Nisida, Tosio. On some probability distributions concerning Poisson process. Math. Japonicae 3, 7–12 (1953).

Let $X(t)$ be a Poisson process with parameter λ and $X(0)=0$. Let $T(n, \omega)=\min\{t; X(t, \omega)=n\}$. For a given t let $T_0(t, \omega)$ and $T_1(t, \omega)$ be the nearest jump points about t. Then $T_0(t, \omega)$ and $T_1(t, \omega)$ are independent and the author finds their distributions. Let $L(i)$ be the time between the ith and the $(i+1)$th jump. Let $M(n)$ and $m(n)$ be the maximum and the minimum of the $L(i_j)$ for a particular choice of n i_j's. It is shown that the distribution of the range $M(n)-m(n)$ is the same as $M(n-1)$ which is in turn given by $\Pr\{M(n-1)<b\}=(1-e^{-\lambda b})^{n-1}$ independent of the particular n i_j's. The distribution of $\frac{1}{2}[M(n)+m(n)]$ and other related distributions are found. *J. L. Snell.*

Gulotta, Beniamino. Sulla estensione della legge di probabilità di Cauchy. Giorn. Ist. Ital. Attuari 16 (1953), 38–50 (1954).

Two different methods for deriving the Cauchy distribution are generalized to the multivariate case. Accordingly

the author obtains two frequency functions:

(a) $f_n(x_1, \cdots, x_n) = C_n[1+(x_1^2+\cdots+x_n^2)/k^2]^{-n}$,
(b) $g_n(x_1, \cdots, x_n) = K_n[1+(x_1^2+\cdots+x_n^2)/k^2]^{-(n+1)/2}$.

The frequency function (b) belongs for all values of n to a stable law, while (a) does not have this property for $n>1$. Therefore (b) appears to be the multivariate analogue to Cauchy's distribution. The frequency function (b) was already discussed by P. Lévy [Théorie de l'addition des variables aléatoires, Gauthier-Villars, Paris, 1937, p. 229] and somewhat earlier by E. Feldheim in a paper [Giorn. Ist. Ital. Attuari 8, 146–158 (1937)] which was not accessible to this reviewer. The author corrects an erroneous determination of the constant K_n given in Feldheim's paper.

E. Lukacs (Washington, D. C.).

Skitovič, V. P. Linear forms of independent random variables and the normal distribution law. Izvestiya Akad. Nauk SSSR. Ser. Mat. 18, 185–200 (1954). (Russian)

Let $X_1, \cdots, X_n$ be mutually independent random vectors. Then, if m linear combinations, with constant coefficients, of these random vectors, are mutually independent, each X_k occurring in at least two different linear combinations is normally distributed. [For this result (which includes all previous results of this type) in the case of random variables, with $m=2$, see a previous paper of the author, Doklady Akad. Nauk SSSR (N.S.) 89, 217–219 (1953); these Rev. 14, 1098.] Closely related theorems are also proved, as well as converses in which each X_k is supposed normally distributed, so that verification of independence of linear combinations becomes a matter of computation.

J. L. Doob (Urbana, Ill.).

Kruskal, Martin D. The expected number of components under a random mapping function. Amer. Math. Monthly 61, 392–397 (1954).

Soit S un ensemble fini de N éléments et f une application de S dans S; un sous-ensemble T de S est invariant si $f^{-1}(T)=T$; les sous-ensembles non vides invariants et minimaux sont les composants de S pour f. En supposant que f est choisie au hasard avec équiprobabilité parmi les N^N applications possibles de S dans S, l'auteur détermine l'espérance mathématique du nombre des composants, l'exprime sous la forme d'une intégrale et étudie son comportement pour $N\to+\infty$.

R. Fortet (Paris).

Henry, Louis. Descendance d'un élément de population. Publ. Inst. Statist. Univ. Paris 1, no. 3, 17–20 (1 plate) (1952).

Let $p(x)$ be the chance that a female will survive to age x and let $m(x)$ be her rate of production of female children per unit of time at that age. If $g(x)\equiv p(x)m(x)$, the expected number of female births per unit time at time t after the birth of a unique female ancestor will be $b(t)\equiv\sum g^{n*}(t)$, the nth term here being the nth Laplace convolution of g with itself. If $m(x)$ vanishes for small and large x, then the effective number of terms will be finite (but will depend on t). Let ρ be the positive root of $\int_0^\infty e^{-\rho x}g(x)dx=1$ and put $h(x)\equiv e^{-\rho x}g(x)$ and $c(t)\equiv e^{-\rho t}b(t)$. Then (large letters denoting Laplace transforms) $B(s)=G(s)/[1-G(s)]$ and $C(s)=H(s)/[1-H(s)]\sim 1/\mu s$ $(s\to 0)$ if $\mu\equiv\int_0^\infty xh(x)dx$. It is now a classical Tauberian inference that under suitable conditions $c(t)\to 1/\mu$ and $b(t)\sim e^{\rho t}/\mu$ $(t\to\infty)$.

The author sketches a direct approach to these results in which the high-order convolutions are replaced by Gaussian approximations (using the central limit theorem) so that $b(t)$ is approximated by a sum of Gaussian ordinates. The arguments are heuristic but interesting.

D. G. Kendall.

Have

***Gnedenko, B. V., and Kolmogorov, A. N. Limit distributions for sums of independent random variables.** Translated and annotated by K. L. Chung. With an Appendix by J. L. Doob. Addison-Wesley Publishing Company, Inc., Cambridge, Mass., 1954. ix+264 pp. $7.50.

This is a translation of the authors' Predel'nye raspredeleniya dya summ nezavisimyh slučaĭnyh veličin [Gostehizdat, Moscow, 1949; these Rev. 12, 839] with certain alterations and additions as noted below. Because of an error discovered in §32, Theorems 3 and 5 of §32 and property 1 of §36 were omitted; the matter is discussed in Appendix II. In §§46–47 improvements from the Hungarian translation [Akadémiai Kiadó, Budapest, 1951; these Rev. 14, 294] are incorporated. Appendix I by J. L. Doob contains further remarks on some of the topics of Ch. 1. The translator has also corrected minor misprints and annotated the text. He wishes to call attention to the following error: The footnote on p. 16 should read: The symbol $A\backslash B$ denotes the set of points in A but not in B.

Kallianpur, G., and Robbins, H. The sequence of sums of independent random variables. Duke Math. J. 21, 285–307 (1954).

Let S_n be the sum of n independent random variables with a common distribution $F(x)$ and $M_n(a)$ be the number of S_j's for $j=1, \cdots, n$ such that $|S_j|<a$. The limit distribution of $M_n(a)$ was obtained by Chung and Kac [Mem. Amer. Math. Soc. no. 6 (1951); these Rev. 12, 722] when F is a symmetrical stable distribution. Let $h(x)$ be Riemann integrable in a finite interval and 0 outside. Limit distributions for $\sum_1^n h(S_j)$ are obtained assuming asymptotic forms for Pr $(a<S_n<b)$. These forms hold for certain absolutely continuous $F(x)$ belonging to the domains of attractions of symmetrical stable distributions, according to a slight modification of an argument of Gnedenko and Kolmogorov [Limit distributions for sums of independent random variables, Gostehizdat, Moscow-Leningrad, 1949; these Rev. 12, 839; see also the translation reviewed above].

K. L. Chung (Syracuse, N. Y.).

Girault, Maurice. Transformation de fonctions caractéristiques par intégration. C. R. Acad. Sci. Paris 238, 2223–2224 (1954).

It is proved that if $\phi(t)$ is a characteristic function (c.f.) then so is $t^{-1}\int_0^t\phi(s)ds$. Perhaps an even simpler proof is to notice that the last may be written $\int_0^1\phi(ut)du$ so that it is the c.f. of the distribution function $\int_0^1F(x/u)du$. Khintchine proved [Izvestiya Naučno-Issled. Inst. Mat. Meh. Tomsk. Gos. Univ. 2, no. 2, 1–7 (1938)] that a distribution function is unimodal if and only if its c.f. can be written in the form $t^{-1}\int_0^t\phi(s)ds$, where ϕ is a c.f. [see Gnedenko and Kolmogorov, Limit distributions for sums of independent random variables, Gostehizdat, Moscow-Leningrad, 1949, p. 169 (or p. 160 of the translation reviewed second above); these Rev. 12, 839].

K. L. Chung (Syracuse, N. Y.).

Skorohod, A. V. On a theorem relative to stable distributions. Uspehi Matem. Nauk (N.S.) 9, 2(60), 189–190 (1954). (Russian)

The analytic nature of stable distribution functions with characteristic exponent $\alpha\geq 1$ is known [see Gnedenko and

Kolmogorov, Limit distributions for sums of independent random variables, Gostehizdat, Moscow-Leningrad, 1949, p. 197 (or p. 183 of the translation reviewed third above); these Rev. 12, 839]. It is here shown that if $\alpha<1$ then the density function is of the form $x^{-1}\phi_1(x^{-\alpha})$ for $x>0$ and $x^{-1}\phi_2(|x|^{-\alpha})$ for $x<0$ where the two ϕ's are entire functions.

K. L. Chung (Syracuse, N. Y.).

Hsu, P. L. A lemma on the coefficient of reduction of a sum of independent variates. Acad. Sinica Science Record 4, 197–200 (1951). (Chinese summary)

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K. L. Chung (Syracuse, N. Y.).

Krein, M. G. On a basic approximation problem of the theory of extrapolation and filtration of stationary random processes. Doklady Akad. Nauk SSSR (N.S.) 94, 13–16 (1954). (Russian)

Let $\sigma(\lambda)$, $-\infty<\lambda<\infty$, be a non-decreasing function such that $\int_{-\infty}^{\infty}d\sigma(\lambda)/(1+\lambda^2)<\infty$. Let Λ_∞ be the Hilbert space of functions $F(\lambda)$, $-\infty<\lambda<\infty$, with inner product $(F,G)=\int_{-\infty}^{\infty}F\bar{G}d\sigma(\lambda)$. Let $I=(a,b)$, $-\infty\leqq a<b<\infty$, be an interval on $-\infty<t<\infty$. Let Λ_I be the closed linear subspace of Λ_∞ determined by $[\exp(it_2\lambda)-\exp(it_1\lambda)]/(i\lambda)$ where $t_1, t_2\in I$. (A) When does $\Lambda_I=\Lambda_\infty$? (B) If $\Lambda\neq\Lambda_\infty$, what is an analytic form of P_IF, the projection of an element $F\in\Lambda_\infty$ onto Λ_I? Let $I=(-a,a)$ and denote Λ_I and P_I by Λ_a and P_a. Let $\sigma(\lambda)$ be an odd function and $\tau(\lambda)=2\sigma(\lambda^{1/2}-0)$, $0<\lambda<\infty$. In the terminology of the author, $\tau(\lambda)$ is the spectral function associated with a mass distribution $M(x)$ on $[0,x)$, $0<x<l$. If

$$t(x)=\int_0^x\sqrt{M'(s)}ds\leqq a\quad(0<x\leqq l),$$

and if, for any $x=x_a$ for which $t(x)=a$, $M(x_a)=M(l)$, then $\Lambda_a=\Lambda_\infty$. The projection P_a is obtained explicitly by means of the expansion theorem associated with

$$\phi(x;\lambda)=1-\lambda\int_0^x(x-s)\phi(s;\lambda)dM(s)\quad(0\leqq x<l).$$

N. Levinson (Cambridge, Mass.).

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Drummond, W. E., and Gardner, C. S. Point source kernel for diffusion with small-angle scattering. Physical Rev. (2) 94, 1491–1496 (1954).

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P. O. Löwdin (Uppsala).

Mathematical Statistics

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D. M. Sandelius (Göteborg).

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L. A. Aroian (Culver City, Calif.).

the author obtains two frequency functions:

(a) $f_n(x_1, \cdots, x_n) = C_n[1+(x_1^2+\cdots+x_n^2)/k^2]^{-n}$,
(b) $g_n(x_1, \cdots, x_n) = K_n[1+(x_1^2+\cdots+x_n^2)/k^2]^{-(n+1)/2}$.

The frequency function (b) belongs for all values of n to a stable law, while (a) does not have this property for $n>1$. Therefore (b) appears to be the multivariate analogue to Cauchy's distribution. The frequency function (b) was already discussed by P. Lévy [Théorie de l'addition des variables aléatoires, Gauthier-Villars, Paris, 1937, p. 229] and somewhat earlier by E. Feldheim in a paper [Giorn. Ist. Ital. Attuari **8**, 146–158 (1937)] which was not accessible to this reviewer. The author corrects an erroneous determination of the constant K_n given in Feldheim's paper.

E. Lukacs (Washington, D. C.).

Skitovič, V. P. Linear forms of independent random variables and the normal distribution law. Izvestiya Akad. Nauk SSSR. Ser. Mat. **18**, 185–200 (1954). (Russian)

Let $X_1, \cdots, X_n$ be mutually independent random vectors. Then, if m linear combinations, with constant coefficients, of these random vectors, are mutually independent, each X_k occurring in at least two different linear combinations is normally distributed. [For this result (which includes all previous results of this type) in the case of random variables, with $m=2$, see a previous paper of the author, Doklady Akad. Nauk SSSR (N.S.) **89**, 217–219 (1953); these Rev. **14**, 1098.] Closely related theorems are also proved, as well as converses in which each X_k is supposed normally distributed, so that verification of independence of linear combinations becomes a matter of computation.

J. L. Doob (Urbana, Ill.).

Kruskal, Martin D. The expected number of components under a random mapping function. Amer. Math. Monthly **61**, 392–397 (1954).

Soit S un ensemble fini de N éléments et f une application de S dans S; un sous-ensemble T de S est invariant si $f^{-1}(T)=T$; les sous-ensembles non vides invariants et minimaux sont les composants de S pour f. En supposant que f est choisie au hasard avec équiprobabilité parmi les N^N applications possibles de S dans S, l'auteur détermine l'espérance mathématique du nombre des composants, l'exprime sous la forme d'une intégrale et étudie son comportement pour $N\to+\infty$.

R. Fortet (Paris).

Henry, Louis. Descendance d'un élément de population. Publ. Inst. Statist. Univ. Paris **1**, no. 3, 17–20 (1 plate) (1952).

Let $p(x)$ be the chance that a female will survive to age x and let $m(x)$ be her rate of production of female children per unit of time at that age. If $g(x)\equiv p(x)m(x)$, the expected number of female births per unit time at time t after the birth of a unique female ancestor will be $b(t)\equiv\sum g^{n*}(t)$, the nth term here being the nth Laplace convolution of g with itself. If $m(x)$ vanishes for small and large x, then the effective number of terms will be finite (but will depend on t). Let ρ be the positive root of $\int_0^\infty e^{-\rho x}g(x)dx=1$ and put $h(x)\equiv e^{-\rho x}g(x)$ and $c(t)\equiv e^{-\rho t}b(t)$. Then (large letters denoting Laplace transforms) $B(s)=G(s)/[1-G(s)]$ and $C(s)=H(s)/[1-H(s)]\sim 1/\mu s$ $(s\to 0)$ if $\mu\equiv\int_0^\infty xh(x)dx$. It is now a classical Tauberian inference that under suitable conditions $c(t)\to 1/\mu$ and $b(t)\sim e^{\rho t}/\mu$ $(t\to\infty)$.

The author sketches a direct approach to these results in which the high-order convolutions are replaced by Gaussian approximations (using the central limit theorem) so that $b(t)$ is approximated by a sum of Gaussian ordinates. The arguments are heuristic but interesting.

D. G. Kendall.

Have

★Gnedenko, B. V., and Kolmogorov, A. N. Limit distributions for sums of independent random variables. Translated and annotated by K. L. Chung. With an Appendix by J. L. Doob. Addison-Wesley Publishing Company, Inc., Cambridge, Mass., 1954. ix+264 pp. $7.50.

This is a translation of the authors' Predel'nye raspredeleniya dya summ nezavisimyh slučaĭnyh veličin [Gostehizdat, Moscow, 1949; these Rev. **12**, 839] with certain alterations and additions as noted below. Because of an error discovered in §32, Theorems 3 and 5 of §32 and property 1 of §36 were omitted; the matter is discussed in Appendix II. In §§46–47 improvements from the Hungarian translation [Akadémiai Kiadó, Budapest, 1951; these Rev. **14**, 294] are incorporated. Appendix I by J. L. Doob contains further remarks on some of the topics of Ch. 1. The translator has also corrected minor misprints and annotated the text. He wishes to call attention to the following error: The footnote on p. 16 should read: The symbol $A\backslash B$ denotes the set of points in A but not in B.

Kallianpur, G., and Robbins, H. The sequence of sums of independent random variables. Duke Math. J. **21**, 285–307 (1954).

Let S_n be the sum of n independent random variables with a common distribution $F(x)$ and $M_n(a)$ be the number of S_j's for $j=1, \cdots, n$ such that $|S_j|<a$. The limit distribution of $M_n(a)$ was obtained by Chung and Kac [Mem. Amer. Math. Soc. no. **6** (1951); these Rev. **12**, 722] when F is a symmetrical stable distribution. Let $h(x)$ be Riemann integrable in a finite interval and 0 outside. Limit distributions for $\sum_1^n h(S_j)$ are obtained assuming asymptotic forms for Pr $(a<S_n<b)$. These forms hold for certain absolutely continuous $F(x)$ belonging to the domains of attractions of symmetrical stable distributions, according to a slight modification of an argument of Gnedenko and Kolmogorov [Limit distributions for sums of independent random variables, Gostehizdat, Moscow-Leningrad, 1949; these Rev. **12**, 839; see also the translation reviewed above].

K. L. Chung (Syracuse, N. Y.).

Girault, Maurice. Transformation de fonctions caractéristiques par intégration. C. R. Acad. Sci. Paris **238**, 2223–2224 (1954).

It is proved that if $\phi(t)$ is a characteristic function (c.f.) then so is $t^{-1}\int_0^t\phi(s)ds$. Perhaps an even simpler proof is to notice that the last may be written $\int_0^1\phi(ut)du$ so that it is the c.f. of the distribution function $\int_0^1 F(x/u)du$. Khintchine proved [Izvestiya Naučno-Issled. Inst. Mat. Meh. Tomsk. Gos. Univ. **2**, no. 2, 1–7 (1938)] that a distribution function is unimodal if and only if its c.f. can be written in the form $t^{-1}\int_0^t\phi(s)ds$, where ϕ is a c.f. [see Gnedenko and Kolmogorov, Limit distributions for sums of independent random variables, Gostehizdat, Moscow-Leningrad, 1949, p. 169 (or p. 160 of the translation reviewed second above); these Rev. **12**, 839].

K. L. Chung (Syracuse, N. Y.).

Skorohod, A. V. On a theorem relative to stable distributions. Uspehi Matem. Nauk (N.S.) **9**, 2(60), 189–190 (1954). (Russian)

The analytic nature of stable distribution functions with characteristic exponent $\alpha\geqq 1$ is known [see Gnedenko and

Kolmogorov, Limit distributions for sums of independent random variables, Gostehizdat, Moscow-Leningrad, 1949, p. 197 (or p. 183 of the translation reviewed third above); these Rev. 12, 839]. It is here shown that if $\alpha<1$ then the density function is of the form $x^{-1}\phi_1(x^{-\alpha})$ for $x>0$ and $x^{-1}\phi_2(|x|^{-\alpha})$ for $x<0$ where the two ϕ's are entire functions.

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L. A. Aroian (Culver City, Calif.).

Gini, Corrado. Estensione della teoria della dispersione e della connessione a serie di grandezze assolute. Giorn. Ist. Ital. Attuari 15, 4–24 (1952).

This is a discussion of measures of dispersion from the viewpoint of descriptive statistics. *E. Lukacs.*

Thompson, H. R. A note on contagious distributions. Biometrika 41, 268–271 (1954).

The author compares the method of obtaining contagious distributions due to Neyman [Ann. Math. Statistics 10, 35–57 (1939)] with that of Darwin [unpublished Ph.D. Thesis, Manchester Univ., 1951] and points out that the latter yields a larger class of distributions which have wider applicability in ecology. *R. P. Peterson.*

Haskey, H. W. A general expression for the mean in a simple stochastic epidemic. Biometrika 41, 272–275 (1954).

The author obtains a polynomial solution for $g_r(x)$ in the differential equation

$$x(1-x)g_r''(x)-n(1-x)g_r'(x)-r(n-r+1)g_r(x)=-f_r(x),$$

where r and n are positive integers ($r\leq n$) and $f_r(x)$ is a polynomial solution to the equation

$$x(1-x)f_r''(x)-n(1-x)f_r'(x)-r(n-r+1)f_r(x)=0.$$

The functions $f_r(x)$ and $g_r(x)$ are used in obtaining a general expression for the mean number of infected individuals in a simple stochastic epidemic. *R. Peterson.*

Birnbaum, Z. W., and Meyer, Paul L. On the effect of truncation in some or all co-ordinates of a multinormal population. J. Indian Soc. Agric. Statistics 5, 17–28 (1953).

As stated in the summary the authors treat the following problem: "Given a p-dimensional normal random variable with means zero, variances one, and correlation matrix R; truncate this random variable in all co-ordinates, say at $t_1, t_2, \cdots, t_p$ respectively, and find expressions for the expectations, variances and covariances of the distribution after truncation."

Using the results obtained, the authors solve the following problems. (1) Given (a) a bivariate normal distribution in X_1, X_2, with expectations 0, variances 1 and correlation coefficient r, and (b) two numbers m_1 and m_2, determine t_1 and t_2 so that, after truncating X_1 at t_1 and X_2 at t_2 the expectations of X_1 and X_2 equal m_1 and m_2 respectively. (2) Given (a) as above and (b) a number m_1, truncate X_1 and X_2 at t_1 and t_2 respectively so that the expectation of X_1 after truncation equals m_1 and the retained part of the population corresponding to the original distribution is as large as possible. *D. M. Sandelius* (Göteborg).

Azorin Poch, Francisco. On the generalized noncentral t distribution and related distributions. Trabajos Estadistica 4, 173–198, 307–337 (1953). (Spanish. English summary).

A review, mainly expository, of the derivation and principal applications of the non-central t distribution. Two results which appear to be new are transformations that convert the non-central t into a variate y which is approximately normally distributed:

$$y_1=n^{1/2}\sinh^{-1}[t/n^{1/2}],\quad y_2=(2n/3)^{1/2}\sinh^{-1}[(3/2n)^{1/2}t].$$

The approximate means and variance y_1 and y_2 are given and the scope of each transformation is discussed. The transform y_2 was previously proposed by Anscombe for the central t-distribution. *W. G. Cochran* (Baltimore, Md.).

Bennett, B. M. Some further extensions of Fieller's theorem. Ann. Inst. Statist. Math. Tokyo 5, 103–106 (1954).

Fieller's theorem is extended to provide confidence limits for the ratio of two normally distributed variates, x and y, for two special cases: (1) the number of observations on $x(n')$ exceed those on $y(n)$, i.e. there are n paired observations plus $(n'-n)$ extra values of x; (2) two sets of paired observations ($n_1<n_2$) are obtained from different sources, with possibly unequal means, variances and covariances. Presumably in case (2), n_1 observations in the two sets are matched randomly or by ordering of the observations.

R. L. Anderson (Raleigh, N. C.).

Have

***Sukhatme, Pandurang V. Sampling theory of surveys with applications.** The Indian Society of Agricultural Statistics, New Delhi, India; The Iowa State College Press, Ames, Iowa, 1954. xxix+491 pp. $6.00.

Chapter headings: 1. Basic ideas in sampling. 2. Basic theory. 3. Stratified sampling. 4. Ratio method of estimation. 5. Regression method of estimation. 6. Choice of sampling unit. 7. Sub-sampling: equal selection probabilities. 8. Sub-sampling: varying selection probabilities. 9. Systematic sampling. 10. Non-sampling errors.

This book is "primarily designed to serve the needs of a text for teaching an advanced course in sampling theory of surveys and of a reference book for statisticians entrusted with the planning of surveys for collecting statistics." Thus it is essentially of the same type as the book by Cochran [Sampling techniques, Wiley, New York, 1953; these Rev. 14, 887] and the books by Hansen, Hurwitz and Madow [Sample survey methods and theory, vols. I, II, Wiley, New York, 1953; these Rev. 15, 332], a main difference being that most of the illustrations of the present book are taken from the agricultural field. The book contains some new results by Indian mathematical statisticians. New contributions by the author include a detailed treatment of a model of non-sampling errors studied recently by the author [Rev. Inst. Internat. Statistique 20, 121–134 (1952); these Rev. 15, 141]. *D. M. Sandelius* (Göteborg).

Scheffé, Henry. Statistical methods for evaluation of several sets of constants and several sources of variability. Chem. Engrg. Progress 50, 200–205 (1954).

Arbey, Louis. Les erreurs expérimentales en chaînes gaussiennes de trois. Bull. Astr. 17, 339–362 (1954).

Let $\{X_i\}$ be a sequence of n measurements of the same unknown quantity θ. The author assumes that the variables of the sequence are jointly normally distributed with unconditional expectations all equal to θ and with unconditional variances all equal to an unknown number σ^2. Furthermore, it is assumed that, given X_{i-2} and X_{i-1}, the variable X_i is independent of the preceding variables $X_1, X_2, \cdots, X_{i-3}$ and that its conditional expectation is a linear function of X_{i-1} and X_{i-2}. The coefficients of this function are unknown except that they do not depend on i. The author deduces maximum likelihood (M.L.) equations for estimating the four unknown parameters and compares the asymptotic variances of the M.L. estimates with those of the estimates based on the method of moments. The moment estimates appear asymptotically equivalent with the M.L. estimates. *J. Neyman* (Berkeley, Calif.).

Strebel, K. Asymptotische Entwicklung einer Summe, die beim Problem der zwei Stichproben auftritt. Math. Ann. 127, 401–405 (1954).

B. L. van der Waerden [Math. Ann. 126, 93–107 (1953); these Rev. 15, 46] proposed a two-sample test based on a statistic X whose variance involves the sum

$$Q=\frac{1}{n}\sum_{k=1}^{n}\Psi^2\left(\frac{k}{n+1}\right),$$

where $\Psi(w)$ is the inverse of the normal distribution function with zero mean and unit variance. The author obtains the asymptotic expansion

$$Q=1-\frac{2}{n}\log n+\frac{1}{n}\log\log n-\frac{1}{n}\log\frac{\pi}{e}+o\left(\frac{1}{n}\right).$$

W. Hoeffding (Chapel Hill, N. C.).

Cox, D. R., and Smith, Walter L. On the superposition of renewal processes. Biometrika 41, 91–99 (1954).

At independent sources random events occur from time to time. The intervals between successive events at any source are assumed to be independent random variables all with the same distribution. The outputs of the sources are combined into one pooled output. Several statistical properties of the pooled output are investigated. For example, the equilibrium variance of the number of events in an interval of length t is found asymptotically for $t\to\infty$ and $t\to 0$. A method is worked out for estimating the number of sources and the coefficient of variation of the original distribution of lengths of intervals. The results are applied to a problem in neurophysiology. *S. W. Nash.*

Wolfowitz, J. Estimation of the components of stochastic structures. Proc. Nat. Acad. Sci. U. S. A. 40, 602–606 (1954).

Two developments of the minimum-distance method are given [see Wolfowitz, Ann. Math. Statistics 25, 203–217 (1954); these Rev. 15, 808]. (1) Let $x_i=u_i+\alpha u_{i-1}$, $i=1, 2, \cdots$, where the u_i are non-constant independent random variables with common unknown distribution $G(z)$ and α is an unknown constant, $|\alpha|<1$; $F(z)$ (unknown) and $F_n(z)$ are respectively the distribution of x_i and the empirical distribution formed from $\lambda_n=(x_1, \cdots, x_n)$. The joint distribution of x_i and x_{i+1} determines α ($|\alpha|<1$) uniquely [the author has stated that line 3 from the bottom of p. 602 should so read]. It is assumed that F satisfies some condition insuring that F determines G uniquely. Let $S=(a, H)$; a is a number, $|a|<1$ and H is a distribution. Let $D(z|S)$ be the distribution of x_i when $\alpha=a$, $G=H$, and let $m(S)=\delta^*(F_n(z), D(z|S))$, where δ^* is the Lévy distance. Let $S^*(\lambda_n)$ be a function of λ_n, satisfying a measurability requirement, such that $m(S^*(\lambda_n))<n^{-2}+\inf_S m(S)$. Then S^* converges with probability 1 to (α, G). (2) Let $x_1=\xi+V_1$, $x_2=\alpha+\beta\xi+V_2$; α and β arbitrary real unknown constants; ξ a random variable with arbitrary unknown non-normal distribution L; V_1 and V_2 jointly normal, independent of ξ, with 0 means and unknown covariance matrix (μ_{ij}). The minimum-distance method gives probability 1 estimators of α and β from observations on independent pairs (x_{1i}, x_{2i}). In general the μ_{ij} and L are not uniquely determined by the distribution of (x_1, x_2) but it follows from work of Reiersøl [Econometrica 18, 375–389 (1950); these Rev. 12, 347] that there is a unique set (μ^0_{ij}, L^0) such that μ^0_{11} and μ^0_{22} are maximal and L^0 has no Gaussian component. Probability 1 estimators can be found for this set. Previous estimators of α and β were given by Neyman [Ann. Math. Statistics 22, 497–512 (1951); these Rev. 13, 481] and Wolfowitz [Skand. Aktuarietidskr. 35, 132–151 (1953); these Rev. 14, 776].

T. E. Harris (Santa Monica, Calif.).

Chanda, K. C. A note on the consistency and maxima of the roots of likelihood equations. Biometrika 41, 56–61 (1954).

Extension of the first part of Huzurbazar's paper [Ann. Eugenics 14, 185–200 (1948), first paragraph; these Rev. 10, 388] to the case of several parameters $\theta_1, \cdots, \theta_r$. Thus, if f_n is the density of n observations, the author is concerned with consistency of a solution of (*) $\partial\log f_n/\partial\theta_i=0$ $(1\leq i\leq r)$ under more stringent conditions than those of Wald [Ann. Math. Statistics 20, 595–601 (1949); these Rev. 11, 261]. The latter proved (e.g.) consistency of any sequence of estimators which make the likelihood at least c times its supremum for some $c>0$, without considering the easier question of when (*) has a solution which maximizes f_n.

J. Kiefer (Ithaca, N. Y.).

Brown, T. M. Standard errors of forecast of a complete econometric model. Econometrica 22, 178–192 (1954).

An econometric model is represented by the linear equation system $\beta Y'+\Gamma Z'=\mu_p'$, where Y, Z and μ_p are respectively vectors of endogenous, predetermined (and normally distributed) disturbance variables and β and Γ are matrices of population parameters. An estimated model $BY'+CZ'=\mu_p'$ is obtained by full information maximum likelihood methods from a sample of T observations. Let Y_F be a conditional forecast of Y given a vector Z_F of predetermined variables, derived from the "forecast reduced" model $Y'=FZ'+u_s'$ with $F=[f_i]=-B^{-1}C$. Then the variance of the forecast from the ith equation is

$$S^2(Y_{iF})=Z_F[S(f_i)]Z_F'+S^2_{si},$$

where $S(f_i)$ is the variance-covariance matrix of the estimates in that equation and S^2_{si} the variance of the corresponding disturbance. Computable expressions for $S(f_i)$ are derived. Corrections for small-sample estimates are suggested but not proved. *H. S. Houthakker.*

Drobot, Stefan, and Warmus, Mieczysław. Dimensional analysis in sampling inspection of merchandise. Rozprawy Mat. 5, 54 pp. (1954). (Russian summary)

The authors set up a "phenomenological" theory of sampling inspection which is not based on probabilistic arguments. The main tool of this theory is dimensional analysis, a brief account of which is given. A variance formula of the type $\sigma^2_{A+B+\cdots}=\sigma_A^2+\sigma_B^2+\cdots$ is included among the postulates of the theory. The authors claim that important problems of sampling inspection can be solved in a simple way by their method. Unknown dimensionless constants occurring in the results obtained are assumed to be determined by suitable experiments. The analysis of the results of such experiments is assumed to be made by methods based on mathematical statistics.

D. M. Sandelius (Göteborg).

Mathematical Economics

★Tustin, Arnold. The mechanism of economic systems. An approach to the problem of economic stabilisation from the point of view of control-system engineering. Harvard University Press, Cambridge, 1953. xi+161 pp. $5.00.

The use of explicit dynamic systems has become quite common in economic theory and in econometrics during the last two decades, but it is still difficult to find good and intelligible introductions to the analysis of such systems, even for the comparatively simple linear case. In electrical engineering, however, linear systems have been applied and studied much more extensively. It was therefore an excellent idea of the author, who is a professor of electrical engineering with a long-standing interest in economic problems, to present this outline of some methods of analysis current in his field as they appear in an economic context. He is particularly concerned with macro-economic stability and its analogies in the electrodynamical theory of control systems.

In his short Chapter I Tustin points out that the maintenance of stability under full employment can only partly be based on empirical models that reflect pre-war conditions, even if the essential non-linearities could be taken into account. He stresses (perhaps even overstresses) the importance of non-linearities in contemporary economic dynamics, but does not promise much help beyond the linear case. Chapter II discusses some simple macro-economic models, their interpretations in engineering terms such as 'feed-back', and the physical systems to which they are equivalent. Chapter III deals with the engineer's approach to the analysis of system behavior. After an enumeration of the characteristic differences between economic and engineering problems it proceeds to an elaborate, mainly diagrammatic treatment of harmonic analysis leading to the Nyquist criterion of stability and its extension for the principal modes of an oscillatory system. The importance of a precise specification of the shape of distributed lags is emphasized. Less attention is paid to "time-series" methods, not only because they are already more familiar to economists, but also because they shed less light on the properties of the solution even though they are more suitable for extrapolation. Graphical methods are used as being more appropriate for dealing with an arbitrary input into the system.

Chapter IV contains a number of suggestions for economic theory. Tustin proposes a quantum theory of investment, according to which the total investment volume is determined by the rate of change and the dispersion of expected profitability for individual investment opportunities. An elaborate scheme of economic interdependence, including non-linear features, is outlined. In accordance with recent work of Hicks and Goodwin little is said about the effect of prices on stability though it would have been interesting to see how prices can be fitted into the dynamical systems previously analyzed.

Chapters V, VI and VII deal mainly with the use of physical analogues, particularly to avoid the iterative methods often necessary in solving non-linear problems. It is not made clear, however, whether analogue computers are superior to the high-speed digital computers that have revolutionized calculating methods.

The principal limitations of this book, namely the summary treatment of non-linearities and the almost complete neglect of random disturbances, are self-imposed and evidently a result of the author's admirable determination to stick to his last. He does not show any of the ignorance of the complexity of economic phenomena which so often distorts the utterances of physicists and engineers on economic topics. On the whole this is a very useful and stimulating work which deserves the serious attention of economists and interested non-professionals alike.

H. S. Houthakker (Stanford, Calif.).

Mathematical Biology

on order

★Statistics and mathematics in biology. Edited by Oscar Kempthorne, Theodore A. Bancroft, John W. Gowen, and Jay L. Lush. The Iowa State College Press, Ames, Iowa, 1954. ix+632 pp. $6.75.

The papers collected in this volume were all contributed to a conference on biostatistics held in June and July, 1952, at Iowa State College. The following is a list of those most likely to be of interest to a mathematical statistician. Sewall Wright: The interpretation of multivariate systems. J. W. Tukey: Causation, regression and path analysis. H. Hotelling: Multivariate analysis. S. L. Isaacson: Problems in classifying populations. K. R. Nair: The fitting of growth curves. C. L. Chiang: Competition and other interactions between species. G. Beall: Data in binomial or near-binomial distribution. J. F. Crow: Breeding structure of populations. The 35 other papers are more concerned with the biological applications, a very wide range of these receiving consideration. The book concludes with a bibliography of 42 pages and a helpful index. *D. G. Kendall.*

Bakan, David. Learning and the principle of inverse probability. Psychol. Rev. 60, 360–370 (1953).

"Science is a way of learning," presumably the best way we know. Some think that a reasonably good model of learning in science is given by the modification of beliefs by observation according to Bayes' theorem. If anyone learns in this way, perhaps animals and people in experimental situations do too—a psychological theory worth exploring.

After such a preamble, with which one may well sympathize, the author enters into theoretical manipulations formally reminiscent of "learning curves" and other empirical data, but of a kind that are not directly confrontable with nature. For example, he calls a function of n defined by

$$P_n = R^n P_0 \{R^n P_0 + (1-P_0)\}^{-1}$$

(which does measure the a posteriori probability of a simple hypothesis after n observations each with likelihood ratio R) a learning curve. But the author does not say how changes in P_n would be reflected in behavior. *L. J. Savage.*

Malécot, G. Les processus stochastiques et la méthode des fonctions génératrices ou caractéristiques. Publ. Inst. Statist. Univ. Paris 1, no. 3, 1–16 (1952).

The author discusses a number of stochastic processes occurring in biology by the method (introduced in congestion theory by C. Palm and developed into a heuristic technique of wide applicability by M. S. Bartlett) of forming a differential equation for the characteristic (or generating) function of the relevant random variables. He first summarises most of the ground covered in three papers by J. E. Moyal, M. S. Bartlett and the reviewer [Symposium

on stochastic processes, Royal Statistical Society, London, 1949; these Rev. **11**, 672] and then turns to an exceedingly interesting treatment of a number of topics in genetics. First he discusses the fluctuations in the gene-frequencies in the absence or presence of selection and/or mutation. He then examines the effect of supposing that the individuals are located in groups isolated save for the inter-group migration following a specified stochastic scheme. This general problem is eventually specialised to one in which mutation (in the absence of selection) is studied in a set of p small equal populations Π_i $(i=1, 2, \cdots, p)$ arranged in a circular pattern, the migration rate from Π_i to Π_j being a function of $(j-i)$ (mod p) alone. Letting $p\rightarrow\infty$ the author then obtains analogous results for a linear array of populations, the migration rate from Π_i to Π_j now depending on $(j-i)$. *D. G. Kendall* (Oxford).

Tricomi, Francesco G. On the statistical distribution of mutant bacteria. Bull. Math. Biophys. **15**, 277–292 (1953).

This is one of a series of papers by the author [these Rev. **13**, 637; **14**, 259, 570, 632] on the statistical distribution of the numbers of (mutant) resistant bacteria formed in a colony growing by binary fission with a fixed division-time and in the absence of "phenotypic delay". It is of interest to compare the results in the present paper with those of Lea and Coulson [J. Genetics **49**, 264–285 (1949)], Harris [Proc. 2nd Berkeley Symposium on Math. Statistics and Probability, 1950, Univ. of California Press, 1951, pp. 305–328; these Rev. **13**, 567], Armitage [J. Roy. Statist. Soc. Ser. B. **14**, 1–40 (1952); these Rev. **14**, 393] and the reviewer [Ann. Inst. H. Poincaré **13**, 43–108 (1952); these Rev. **15**, 243] where different assumptions were made. The problem originated with a famous experiment of Luria and Delbrück [Genetics **28**, 491–511 (1943)], and the chief difficulty is the determination of the form of the distribution when the generating function has been found; it is with this question that the author is mainly concerned. In his problem the observed number of resistant bacteria is taken to be $N\equiv n_1+2n_2+\cdots+kn_k$ where the n_j are independent Poisson variables with mean values $\lambda_1, \lambda_2, \cdots, \lambda_k$, so that the generating function is $\exp\left(\sum\lambda_j z^j-\sum\lambda_j\right)$. The author discusses the problem in this general form although in the application in view $\lambda_j=\lambda/j$ when $j=1, 2, 4, \cdots, 2^{m-1}$ and $\lambda_j=0$ otherwise. After some elementary calculations of moments, etc., he expresses $p_n\equiv\text{prob}\,(N=n)$ as a contour integral by Cauchy's theorem and gives an asymptotic formula for p_n by an appeal to the method of steepest descents. [The reviewer could not see the force of the argument in the second sentence following (16), but a more literal application of the saddle-point method does seem to lead to the required result, (18), if p_n is replaced by a suitable moving average.]

This as it stands is of no use in the bacterial mutation problem without further information on the partial sums of the non-continuable power-series $\sum z^{2^m}$, and to this question the author now turns. He shows that these partial sums can be dominated by using the entire function $G(z)\equiv\sum z^n/[n!(2^n-1)]$, and he investigates the properties of G in great detail, giving asymptotic formulae and tables and graphs from which the form of the distribution $\{p_n\}$ can be inferred. Some of his formulae should greatly simplify the present methods for estimating the mutation-rate; thus the reviewer would like to point out that with the author's notation the transformed variable

$$\xi\equiv(2aA)^{1/2}\exp\left[\frac{m\lambda-n}{2\lambda A}\right]$$

should according to his (39) have approximately a standardized normal distribution with the ordinates increased in the ratio $\sqrt{2}:1$, when ξ is in the interval corresponding to the practically important region $n\ll m\lambda$.

An Italian summary of this paper will be found in the paper reviewed below. *D. G. Kendall* (Oxford).

Tricomi, Francesco G. Un problema di statistica matematica sorto dalla batteriologia. Giorn. Ist. Ital. Atturi **15**, 25–39 (1952).

An account of this work (in English) has also been published in the paper reviewed above. *D. G. Kendall.*

Davis, Robert L. Structures of dominance relations. Bull. Math. Biophys. **16**, 131–140 (1954).

Let $A=(a_{ij})$ be a matrix of 0's and 1's such that $a_{ii}=0$ for every i, and for every i and j, $a_{ij}=1$ or $a_{ji}=1$ but not both. Then A represents a dominance relation. Two relations A and B are isomorphic in case there exists a permutation matrix P such that $B=PAP^{-1}$. The structure of a relation A is the class of all relations isomorphic to A. The author extends results of a previous paper [Proc. Amer. Math. Soc. **4**, 486–495 (1953); these Rev. **14**, 1053] to obtain the number of distinct structures definable on a set of N elements. *A. S. Householder* (Oak Ridge, Tenn.).

TOPOLOGY

Baebler, F. Über den Zerlegungssatz von Petersen. Comment. Math. Helv. **28**, 155–161 (1954).

This is another proof, using alternating circuits, of the theorem of Petersen on the factors of cubic graphs. *W. T. Tutte* (Toronto, Ont.).

Tutte, W. T. A short proof of the factor theorem for finite graphs. Canadian J. Math. **6**, 347–352 (1954).

In an earlier paper [same J. **4**, 314–328 (1952); these Rev. **14**, 67] the author gave a necessary and sufficient condition for a locally finite graph to contain an f-factor. In this paper a shorter proof of this result, applicable to finite graphs; is given. It is based on the author's criterion for the existence of a 1-factor [J. London Math. Soc. **22**, 107–111 (1947), these Rev. **9**, 297]. *G. A. Dirac* (London).

Tutte, W. T. The 1-factors of oriented graphs. Proc. Amer. Math. Soc. **4**, 922–931 (1953).

The paper is chiefly concerned with a topic in the theory of oriented graphs which is the analogue of the concept of factors of unoriented graphs. If n is any positive integer, an n-factor of an oriented graph G is defined to be a subgraph of G which includes all the vertices of G and in which each vertex is the positive end of exactly n edges and also the negative end of exactly n edges. The chief result obtained is a necessary and sufficient condition that a given locally finite oriented graph without loops shall have a 1-factor. A graph is said to be locally finite if each vertex is associated with at most a finite number of edges.

If S is a subset of the set V of vertices of the oriented graph G, the graph obtained from G by deleting all vertices

in S, and all edges having a vertex in S as an end, is denoted by G_S. Such a subset S of V is called independent in G if there is no finite oriented path in G whose origin and terminus are members of S. If in addition there is no pair of infinite oriented paths in G, one having a vertex in S as origin and the other having a vertex in S as terminus, then S is said to be strictly independent in G. The following theorem is proved: A locally finite oriented graph G which contains no loops has no 1-factor if and only if there exist disjoint finite sets S and T of vertices of G such that T is strictly independent in G_S and S contains fewer vertices than T.

An application to unoriented graphs is also given. A subgraph of an unoriented graph Γ which includes all the vertices of Γ and in which each connected component is regular of degree 1 or regular of degree 2 is called a Q-factor of Γ. The following result is established: A locally finite unoriented graph Γ without loops has no Q-factor if and only if there are disjoint finite sets U and V of vertices of Γ such that U contains fewer vertices than V and each edge of Γ which has one end in V has the other end in U. For a locally finite even graph this is equivalent to the theorem of P. Hall and R. Rado on the factorization of even graphs.
G. A. Dirac (London).

Rédei, Ladislaus. Über die Kantenbasen für endliche vollständige gerichtete Graphen. Acta Math. Acad. Sci. Hungar. **5**, 17–25 (1954). (Russian summary)

A Kantenbasis is an oriented graph G such that for any two distinct vertices a and b of G there exists an oriented path in G leading from a to b, and another leading from b to a, and such that this property is lost when any edge is removed. The author shows how the Kantenbasis of $n+1$ vertices can be determined from those of $\leqq n$ vertices. He shows also that no Kantenbasis has a Thomsen graph or complete 5-graph as a subgraph. From the latter result he concludes, erroneously, that every Kantenbasis is planar. (To construct a non-planar Kantenbasis take a Thomsen graph, subdivide each edge by taking the mid-point as a new vertex, select a vertex V of degree 3 in the resulting graph and direct each edge away from V. Finally join the two vertices distant four edges from V to V by two new edges directed to V.) *W. T. Tutte* (Toronto, Ont.).

Ringel, Gerhard. Bestimmung der Maximalzahl der Nachbargebiete auf nichtorientierbaren Flächen. Math. Ann. **127**, 181–214 (1954).

If S is a surface, let $k(S)$ be the positive integer such that a map whose countries cannot be coloured with fewer than $k(S)$ colours (in such a way that no two countries which have a common frontier are coloured alike) can be drawn on S, but no map whose countries cannot be coloured in this way with fewer than $k(S)+1$ colours can be drawn on S. P. J. Heawood [Quart. J. Pure Appl. Math. **24**, 332–338 (1890)] proved that if S has connectivity h,

$$k(S)\leqq H(h)=[3\tfrac{1}{2}+\tfrac{1}{2}\sqrt{(24h-23)}].$$

He conjectured that, if S is a closed surface, $k(S)=H(h)$, so that in this case the upper bound $H(h)$ is best possible. In this paper it is proved that for a closed one-sided surface the upper bound is actually attained, by showing that if $h\neq 1$ and $h\neq 3$ then a map consisting of $H(h)$ mutually adjacent countries can always be drawn on a closed one-sided surface of connectivity h. It is also proved that if $24r^2+6r+3\leqq h\leqq 24r^2+10r+1$ $(r=1, 2, \cdots)$ then a map consisting of $H(h)$ mutually adjacent countries can be drawn on a closed two-sided surface of connectivity h.

For the remaining closed two-sided surfaces it has been shown that Heawood's upper bound is best possible only for $h=3, 5, 7, 9, 11, 13, 15, 17, 19$. The reviewer has however shown that if S is any surface whatever, it can be determined in a finite number of steps whether or not $k(S)>7$, and that, if $k(S)>7$, $k(S)$ can be determined in a finite number of steps [J. London Math. Soc. **28**, 476–480 (1953); these Rev. **15**, 144]. *G. A. Dirac* (London).

Ringel, Gerhard. Lokal-reguläre Zerlegungen geschlossener orientierbarer Flächen. Math. Z. **59**, 484–495 (1954).

If m, r and s are positive integers, a closed polyhedron is said to be locally regular of type (m, r, s) if there are m faces, each face has r edges, and s edges meet in each corner. Two faces may have more than one edge in common, and a face may border on itself. A closed surface has characteristic $\geqq -2$, so that the following are necessary conditions for the existence of a locally regular polyhedron of type (m, r, s): $(rs-2r-2s)m/2s$ is integral and $\geqq -2$; mr is even. However, these conditions are not sufficient; to show this the example $m=2$, $r=9$, $s=3$ is referred to. But if only orientable surfaces are considered, it is thought that these two conditions are sufficient, according to the conjecture: A locally regular polyhedron of type (m, r, s) can be constructed on a closed orientable surface of genus p if and only if $m(rs-2r-2s)=4s(p-1)$ and mr is even. In this paper the conjecture is proved in the following three important special cases: (i) $p=2$, (ii) $s=3$, (iii) $m=1$. (In the case of the sphere the conjecture is known to be true, for $r\geqq 3$ it gives the Platonic solids.) *G. A. Dirac.*

Dirac, G. A., and Schuster, S. A theorem of Kuratowski. Nederl. Akad. Wetensch. Proc. Ser. A. **57** = Indagationes Math. **16**, 343–348 (1954).

The authors simplify the proof of Kuratowski's theorem [Fund. Math. **15**, 271–283 (1930)] that a finite graph is planar if and only if it has no subgraph which can be obtained from a Thomsen graph or complete 5-graph by subdividing the edges. They discuss the possibility of extending the theorem to countable infinite graphs. *W. T. Tutte.*

Smirnov, Yu. On completeness of uniform spaces and proximity spaces. Doklady Akad. Nauk SSSR (N.S.) **91**, 1281–1284 (1953). (Russian)

Let P_Σ denote a proximity space ("δ-space") P with a uniformity Σ. A family ξ of subsets of P_Σ is called a $\Sigma\delta$-collection if it contains arbitrarily small sets and if any $A\in\xi$ is a δ-neighborhood of some $B\in\xi$ [cf. Smirnov, Mat. Sbornik N.S. **31**(73), 543–574 (1952); these Rev. **14**, 1107]. The set ΣP of all maximal centred $\Sigma\delta$-collections becomes, in a natural way, a δ-space (containing the given space P_Σ). The properties of ΣP are considered; it is stated that the above construction of ΣP works, too, if Σ is a "pseudo-uniformity" (i.e. if the intersection of two entourages is not required to be an entourage).

Let Σ_a denote, for a given δ-space P, the pseudo-uniformity determined by the family of all uniform δ-coverings; then $\Sigma_a P$ is a completion of P [cf. Smirnov, Doklady Akad. Nauk SSSR (N.S.) **88**, 761–764 (1953); these Rev. **15**, 144]. It is not known whether Σ_a is a uniformity; as the author states, his counterexample [loc. cit., p. 764] is not correct. In the final section of the note, the author considers relations between uniform and proximity spaces generated by a topological group. *M. Katětov* (Prague).

Nakano, Hidegorô. On completeness of uniform spaces. Proc. Japan Acad. **29**, 490–494 (1953).

L'auteur démontre un certain nombre de critères pour qu'un espace uniforme E soit complet lorsque sa structure uniforme est définie par la condition d'être la moins fine rendant uniformément continues des applications de E dans des espaces uniformes complets S_λ, ou la moins fine rendant uniformément équicontinues des familles d'applications de E dans les S_λ. Il en déduit des critères analogues pour certaines topologies sur des espaces d'applications linéaires d'un espace vectoriel dans un espace vectoriel topologique.

J. Dieudonné (Evanston, Ill.).

Ward, L. E., Jr. Partially ordered topological spaces. Proc. Amer. Math. Soc. **5**, 144–161 (1954).

A partially ordered (respectively, quasi ordered) topological space, abbreviated POTS (QOTS), is a topological space with a partial (quasi) order such that the sets $\{y|y\leqq x\}$, $\{y|y\geqq x\}$ are closed for any x. After a few preliminary results, the author considers QOTS such that every maximal chain is compact (c.m.c. condition). Some results: the c.m.c. condition implies that any order-dense maximal chain is connected; it is satisfied if, and only if, max C and min C are compact non-void, for any closed chain $C\neq 0$ (max C, min C denotes, respectively, the set of maximal, minimal elements of C); if X is an order dense POTS, and either max C or min C is connected, then the c.m.c. condition implies that X is connected.

In the last section of the note, fixed-point theorems are proved for QOTS, e.g., if X is a Hausdorff QOTS satisfying the c.m.c. condition, $f: X\to X$ is continuous and preserves the order, and some x is comparable with $f(x)$, then there exists $K\subset X$ with $K\neq 0$, $f(K)=K$, $x\leqq y\leqq x$ for any $x, y\in K$. As an important application, the following result is obtained: suppose X is a locally connected continuum, $E\subset X$ is an end element, $f: X\to X$ is monotone, $f(E)=E$; then there exists a continuum $K\subset X$, $K\neq 0$ such that $f(K)=K$, and either $K\subset X-E$ or K is a cut-point [cf. G. E. Schweigert, Amer. J. Math. **66**, 229–244 (1944); these Rev. **5**, 274; A. D. Wallace, Bull. Amer. Math. Soc. **51**, 413–416 (1945); these Rev. **6**, 278].

N. Katětov (Prague).

Ogasawara, Tôzirô, and Funakoshi, Junzô. On topological operations determined by local characters. J. Sci. Hiroshima Univ. Ser. A. **15**, 103–112 (1951).

In a topological space R let P_ϕ be an arbitrary family of subsets including 0. For each A the ϕ-closure A^ϕ is defined to be the complement of $\bigcup\{G\}$ for all open sets G with GA in P_ϕ. This ϕ-closure is always a closed subset of the ordinary closure and coincides with ordinary closure if P_ϕ consists of 0 only. A detailed analysis is made of the ϕ-closure, generalizing many theorems for ordinary closure. Special assumptions are also studied: (H) $A\in P_\phi$; $B\subset A$ implies $B\in P_\phi$; $(\mathrm{H})_\omega$ $A\in P_\phi$, B open in A implies $B\in P_\phi$; (A) A, $B\in P_\phi$ imply $A\cup B\in P_\phi$; $(\mathrm{A})_\omega$ A, $B\in P_\phi$ and of them closed in $A\cup B$ imply $A\cup B\in P_\phi$.

I. Halperin (Utrecht).

Iséki, Kiyoshi. A note on normal spaces. Math. Japonicae **3**, 45 (1953).

This proves that countably many closed sets which form a discrete collection in a normal space can be enclosed in disjoint open sets. [This theorem is known; see Kuratowski, Fund. Math. **24**, 259–268 (1935), p. 260.]

A. H. Stone.

Iséki, Kiyoshi. A note on hypocompact spaces. Math. Japonicae **3**, 46–47 (1953).

This proves that the product of an S-space (=hypocompact space) and a compact space is an S-space. [This theorem is known; see Begle, Bull. Amer. Math. Soc. **55**, 577–579 (1949); these Rev. **10**, 726.]

A. H. Stone.

Floyd, E. E., and Klee, V. L. A characterization of reflexivity by the lattice of closed subspaces. Proc. Amer. Math. Soc. **5**, 655–661 (1954).

Let the lattice of closed subspaces of a normed space B be given the order topology [G. Birkhoff, Lattice theory, Amer. Math. Soc. Colloq. Publ., vol. 25, rev. ed., New York, 1948; these Rev. **10**, 673]. Then L is a T_1-space; the authors prove: L is a Hausdorff space if and only if P is reflexive.

M. M. Day (Urbana, Ill.).

Eisenstadt, B. J. The space of point homotopic maps into the circle. Trans. Amer. Math. Soc. **77**, 62–85 (1954).

Let R_{2q} denote the group of real numbers modulo $2q$ $(q\geqq 1)$ with the invariant metric taken so that the distance $\rho(a)$ from $a\in R_{2q}$ to the identity is the least absolute value of the numbers in the coset a. The set of all continuous functions from a topological space X into R_{2q} forms a complete metric group if functions are added pointwise and the distance from f to the identity is taken as $\sup_{x\in X}\rho(f(x))$. The author studies the closed subgroup, $R_{2q}(X)$, consisting of the functions that are homotopic to constants. The investigation is carried out along the lines of the existing theory of the Banach space, $C(X)$, of continuous real-valued functions, culminating in an analogue of the Banach-Stone theorem: If X and Y are compact Hausdorff spaces, and $R_{2q}(X)$ is isometrically isomorphic to $R_{2q'}(Y)$, then $q=q'$ and X is homeomorphic to Y. A characterization of $R_{2q}(X)$ as a metric abelian group is obtained for compact connected X. The machinery for doing this consists, in part, of constructing $C(X)$ out of $R_{2q}(X)$. Some new results concerning $C(X)$ are also obtained.

M. Jerison (Lafayette, Ind.).

Tominaga, Akira. On extensions of a metric. J. Sci. Hiroshima Univ. Ser. A. **17**, 185–191 (1953).

The author proves two similar theorems on extensions of metrics. The first of these is as follows. If the space R is regular and has a countable base and F a closed metric subset of R which is isometric to the euclidean n-cube Q^n, then there is a bounded metric on R which agrees with the euclidean metric on F. The proof depends on embedding R, analogously to the classical metrization theorems, in $Q^n\times Q^\omega$, where Q^ω is the Hilbert cube.

M. E. Shanks.

Etter, D. O., and Griffin, John S., Jr. On the metrizability of the bundle space. Proc. Amer. Math. Soc. **5**, 466–467 (1954).

If $\{X, B, \pi, Y, \mathfrak{V}, \phi, G\}$ is a fibre bundle whose base space B and fibre Y are metrizable, then the bundle space X is also metrizable.

E. Hewitt (Seattle, Wash.).

Jones, F. Burton. On a property related to separability in metric spaces. J. Elisha Mitchell Sci. Soc. **70**, 30–33 (1954).

The author gives the construction of a metric space S in which the closure of each countable set is countable without S being the union of countably many discrete sets. S is obtained by metrization of an appropriate model of an Aronszajn tree [cf. Jones, same J. **69**, 30–34 (1953); these Rev. **15**, 18]. In connection with Alexandroff's results

[Math. Ann. **92**, 294–301 (1924)] and his own [Bull. Amer. Math. Soc. **41**, 437–439 (1935)], the author formulates the following problem: Is every connected locally peripherally separable metric space separable? *G. Kurepa.*

Grace, Edward E. A note on linear spaces and unicoherence. J. Elisha Mitchell Sci. Soc. **70**, 33–34 (1954).

Eilenberg [Amer. J. Math. **63**, 39–45 (1941); these Rev. **2**, 179] has characterized linear intervals as non-degenerate, connected, locally connected, separable topological spaces X for which $X \times X$ is disconnected by its diagonal. This note first remarks that if X is a Moore space, the requirement of separability can be omitted; this follows from work of Eilenberg (loc. cit.) and F. B. Jones [Bull. Amer. Math. Soc. **45**, 623–628 (1939); these Rev. **1**, 45]. Secondly, a simple proof is given of the fact that if X is a non-unicoherent connected topological space (the author places further restrictions on X but does not use them), then X contains 3 points no one of which separates the other two. *A. H. Stone* (Manchester).

Skornyakov, L. A. Topological projective planes. Trudy Moskov. Mat. Obšč. **3**, 347–373 (1954). (Russian)

A topological projective plane P is defined as follows: both the space of points and the space of lines are endowed with a Hausdorff topology, and both operations of incidence are assumed to be (jointly) continuous. Even if one assumes compactness and connectedness there are a great variety of possible non-Desarguian planes, and so the project of classifying all topological projective planes is not undertaken. However, there are many interesting partial results. For instance: the space of points of P is either connected or totally disconnected; it is either discrete or dense in itself; if it is locally compact it is compact.

The author also studies topological partial planes, notably Euclidean planes where a line is deleted (these are more often called affine planes), co-Euclidean planes where a point is deleted, and dual Euclidean planes where both are deleted. Dual Euclidean planes are the natural object in which to introduce Marshall Hall's ternary operation. If suitable additional continuity assumptions on the plane are made, it turns out that the ternary operation is continuous in all desirable senses. Conversely, a topological ternary operation defines a topological dual Euclidean plane. The question as to whether such a plane can be embedded in a topological projective plane is left open in general, but is answered in the affirmative in the locally compact connected case. *I. Kaplansky* (Chicago, Ill.).

Hemmingsen, E. Plane continua admitting non-periodic autohomeomorphisms with equicontinuous iterates. Math. Scand. **2**, 119–141 (1954).

Let M be a plane continuum and let f be a homeomorphism of M onto M such that the set of all iterates of f is equicontinuous. The following theorems are proved. (1) Every component of int M contains a simple closed curve which is invariant under some positive iterate of f. (2) If K is a simple closed curve contained in int M and if both K and some point of K are invariant under a positive iterate of f, then f is periodic on the component of int M which contains K. (3) If W is a component of int M and if f is nonperiodic at some point of W, then $\bar{W}=M$, M is topologically a circular disc or an annulus, and f or f^2 is topologically an irrational rotation. (4) If f is pointwise periodic on some component of int M, then f is pointwise periodic on int M. Some examples are discussed. Remarks on the analogous situation in 3-space conclude the paper.
W. H. Gottschalk (Philadelphia, Pa.).

Williams, R. F. Local contractions of compact metric sets which are not local isometries. Proc. Amer. Math. Soc. **5**, 652–654 (1954).

Let X be a compact metric space, let f be a mapping of X onto X, and let f be a local contraction, that is, for each $x \in X$ there exists $\mu>0$ such that $y \in X$ with $\rho(x, y)<\mu$ implies $\rho(fx, fy) \leqq \rho(x, y)$. The author constructs examples to show that f need not be a local isometry even though, for example, X is connected or X is totally disconnected or f is homeomorphic. This answers a question raised by Edrei [Proc. London Math. Soc. (3) **2**, 272–278 (1952); these Rev. **14**, 305]. *W. H. Gottschalk* (Philadelphia, Pa.).

Noguchi, Hiroshi. A characterization of homotopically labil points. Kōdai Math. Sem. Rep. **1954**, 13–16 (1954).

A point P of a space M is called homotopically labile if for each neighborhood U of P there is a deformation $f(x, t)$ of M such that (1) $f(x, t)=x$, $0 \leqq t \leqq 1$, $x \in M-U$; (2) $f(x, t) \in U$, $0 \leqq t \leqq 1$, $x \in U$; (3) $f(x, 1) \neq P$, $x \in M$. For finite-dimensional locally finite complexes, it is shown that a vertex P is homotopically labile if and only if there is a star, C, of P such that $C-(P)$ is contractible. A consequence of this is that in a homogeneous 2-dimensional complex the existence of a homotopically labile point is equivalent to the existence of a free side. An example shows that this last result does not hold in higher dimensions. *E. G. Begle.*

Keldyš, Lyudmila. Example of a one-dimensional continuum with a zero-dimensional and interior mapping onto the square. Doklady Akad. Nauk SSSR (N.S.) **97**, 201–204 (1954). (Russian)

There is constructed a one-dimensional locally connected continuum X and a light open map f of X onto a square C. Here C is a square in xy-space while X is constructed in two stages, the final stage yielding X as a set in $xyzt$-space which projects onto C in a light open fashion under the natural projection. *E. E. Floyd* (Charlottesville, Va.).

Cronin, Jane. Topologic degree of some mappings. Proc. Amer. Math. Soc. **5**, 175–178 (1954).

Assuming no fixed points on the boundary, let T be the transformation of the n disc to the n disc of the Euclidean space E, defined by a power series in $x_1, \cdots, x_n$, with real coefficients. The degree of this transformation is related to that of the transformation of the complex spheres obtained by interpreting $x_1, \cdots, x_n$ as complex variables. More precise bounds are obtained for the more homogeneous polynomials and a full treatment is given when $n=2$.
D. G. Bourgin (Rome).

Fort, M. K., Jr. Open topological disks in the plane. J. Indian Math. Soc. (N.S.) **18**, 23–26 (1954).

A set X is said to have the almost-fixed-point property (AFPP) if for each mapping f of X into itself and for each number $\epsilon>0$, there is a point $x \in X$ with $\rho(f(x), x)<\epsilon$. A set A is called an ϵ-retract of a set B if there is a retraction f of B onto A such that $\rho(f(x), x)<\epsilon$ for all x in B. A set G which is homeomorphic to the set D of all complex numbers z such that $|z|<1$ is called an open topological disk. With these definitions in mind, consider the following three properties of an open topological disk G in the plane. (I) There is a uniformly continuous homeomorphism of D

onto G. (II) For every $\epsilon>0$, there is a closed topological disk which is an ϵ-retract of G. (III) G has the AFPP.

It is shown that (I) implies (II) and (II) implies (III). Then it is shown that if the boundary of G is locally connected, G has property (I) and hence has the AFPP. Examples are given to show that (I) does not follow from (II) nor (II) from (III) and that not all open topological disks have the AFPP. *E. G. Begle* (New Haven, Conn.).

Koseki, Ken'iti. Über die Abbildungen von mehrdimensionalen einfach-zusammenhängenden Gebieten auf Kugeln und ihre Begrenzungen. I. Jap. J. Math. 22 (1952), 87–100 (1953).

The author proves the following theorem (in slightly different terminology). Suppose that R is a bounded set in Euclidean 3-space E^3 which is the common boundary of a bounded region G and an unbounded region G_1, where R is simply accessible from G. Suppose there exists a homeomorphism f of the interior H of the unit sphere onto G such that f has a continuous extension sending $\bar{H}$ into $\bar{G}$. Then there exists a homeomorphism g of $\bar{H}$ onto $\bar{G}$. In the above, R is simply accessible from G if for each $a \in R$ and each sequence (a_i) in G tending to a, there exists an arc in $G+a$ which contains all the a_i. A somewhat similar result has been obtained independently by Floyd and Fort [Proc. Amer. Math. Soc. 4, 828–830 (1953); these Rev. 15, 244].

E. E. Floyd (Charlottesville, Va.).

Fet, A. I. Generalization of a theorem of Lyusternik-Šnirel'man on coverings of spheres and some theorems connected with it. Doklady Akad. Nauk SSSR (N.S.) 95, 1149–1151 (1954). (Russian)

The present paper generalizes published results on antipodal maps of spheres to arbitrary maps of period 2. Theorem 1: If θ is a map of S^n to itself of period 2 and if $\{F_1, F_2, \cdots, F_{n+1}\}$ is a covering of S^n by closed sets, then at least one of the sets F_i contains a pair of points a, θa.

Let Π^n be the space obtained from S^n by identifying a, θa, for all $a \in S^n$, and let $\hat{\Pi}^{n+1}$ be the space obtained from T^{n+1}, a ball bounded by S^n, by making the same identifications. The proof of Theorem 1 proceeds by defining a deformation D_t of the identity map of S^n such that $D_1\theta=\tau D_1$, where τ is the antipodal map. (The deformation D_t is defined if θ is without fixpoints, which is the non-trivial case.) By means of D_1 one may define a map f: $\hat{\Pi}^{n+1}$, $\Pi^n \to p^{n+1}$, p^n, where p^r is projective r-space, which induces an isomorphism of $H_{n+1}(\hat{\Pi}^{n+1}, \Pi^n)$ onto $H_{n+1}(p^{n+1}, p^n)$. If V^1 is the basic 1-cocycle (mod 2) of p^{n+1} and $Z^{n+1}=(f^*V^1)^{n+1}$, then Z^{n+1} non-~ 0 in $\hat{\Pi}^{n+1}$, whence (Theorem 2) the category of $\hat{\Pi}^{n+1}$ is $(n+2)$; from this Theorem 1 is an easy consequence. (The author observes (Theorem 3) that one proves in the same way that the category of Π^n is $(n+1)$.)

From Theorem 1 the author derives a generalization of a theorem due to Borsuk, namely, Theorem 4: If Φ is a vector field on T^{n+1} without null vectors on S^n and if the vectors at a, θa have different directions for all $a \in S^n$, then Φ has a null vector on T^{n+1}. From this Theorem 5, generalizing the Borsuk-Ulam theorem, follows immediately: If f is a map of S^n to Euclidean n-space, then $fa=f\theta a$ for some $a \in S^n$.

P. J. Hilton (Cambridge, England).

Kinoshita, Shin'ichi. Notes on some theorems on the sphere. Proc. Japan Acad. 29, 548–549 (1953).

Using known theorems about antipodic mappings of spheres, the author shows that (1) a mapping of a sphere S^n onto itself which is of even degree maps at least one antipodic pair onto a single point; (2) if $F_0, \cdots, F_{n+1}$ are $n+1$ closed sets in S^n with empty intersection, there is at least one point p in S^n such that $p \in F_i$ if and only if the antipode of p_i is in F_i; (3) the same holds for n closed sets with no hypothesis made about their intersection. *P. A. Smith.*

Weier, Josef. Fixpunktmindestzahlen in unendlichen Polyedern. Math. Ann. 127, 319–339 (1954).

The main theorem of this paper is the following. Let K be a connected infinite simplicial complex having the property that no vertex is a cut point of its closed star. Let f be a mapping of K into itself. Then there is a mapping f' which is homotopic to f and which has no fixed point. As a refinement of this theorem, it is shown that if f is the identity mapping, then f' can be chosen to be arbitrarily close to f.

E. G. Begle (New Haven, Conn.).

Weier, Josef. Normale Abbildungsscharen. Math. Z. 59, 356–374 (1954).

The first theorem proved here is the following: Let P be a finite simplicial complex, with the property that no vertex is a cut point of its closed star, and let f be the identity map of P into itself and ϵ a positive number. Then there are points $q_1, q_2, \cdots, q_m$ in P and a homotopy $\{f_t\}$ such that $f_0=f$, $\rho(f, f_t)<\epsilon$ for $0\leq t\leq 1$, and such that, for $0<t\leq 1$, the points $q_1, q_2, \cdots, q_m$ are the fixed points of the mapping f_t.

Various refinements of this are also proved. The following is typical: Let P, f, and ϵ be as above. Then there exist curves $\{q_{i_t}\}$, $0\leq t\leq 1$, $1\leq i\leq m$, and a homotopy $\{f_t\}$ such that $f_0=f$, $\rho(f, f_t)<\epsilon$ and in addition the distinct points $q_{1t}, q_{2t}, \cdots, q_{mt}$ are the fixed points of the mapping f_t for $0<t<1$, while f_1 has just one fixed point, $q_{11}=q_{21}=\cdots=q_{m1}$.

E. G. Begle (New Haven, Conn.).

Weier, Josef. Über normale Abbildungsscharen. Math. Nachr. 11, 219–230 (1954).

Let f', f'' be mappings $S\to S$ and F a continuous family of mappings f_t: $S\to S$, $0\leq t\leq 1$, such that $f_0=f'$, $f_1=f''$. Suppose f', f'' are fixed-point free. It is shown that if S is an euclidean space of dimension >1 there exists an F such that each f_t is fixed-point free. If S is a sphere of dimension >1 and f', f'' are homotopic to the identity, there exists an F such that $f_{1/2}$ has a single fixed point, the other f_t's being fixed-point free. Suppose f', f'' each have a single fixed point. If S is a sphere of dimension >0, and f', f'' are homotopic to the identity, there exists an f such that each f_t has a single fixed point p_t depending continuously on t; the path defined by p_t can be pre-assigned. *P. A. Smith.*

Nikaido, Hukukane. Zusatz und Berichtigung für meine Mitteilung "Zum Beweis der Verallgemeinerung des Fixpunktsatzes" in diesen Reports, Bd. 5, Nr. 1, 1953. Kōdai Math. Sem. Rep. 1954, 11–12 (1954).

Complements to an earlier paper [same Rep. 1953, 13–16; these Rev. 15, 52], including a proof of the von Neumann-Kakutani theorem for the finite-dimensional case as well corrections of typographical errors. *E. G. Begle.*

Noguchi, Hiroshi. On isotopy. I. Tôhoku Math. J. (2) 5, 104–108 (1953).

It is shown that if A is an n-dimensional compactum and M a combinatorial manifold, with dim $M\geq 2n+1$, then for each $\epsilon>0$ and each mapping f of A into M, there is a homeomorphism g of A into M with $\rho(f, g)<\epsilon$. Using this

result, it is shown that if dim $M \geqq 2n+3$, then the isotopy and homotopy classes of maps of A into M are in one-to-one correspondence. By an isotopy of two maps f_0 and f_1 is meant a mapping F of $A\times I$ into M such that $F|Ax(0)=f_0$, $f|Ax(1)=f_1$ and $F|Ax(t)$ is a homeomorphism for each t, $0\leqq t\leqq 1$. *E. G. Begle* (New Haven, Conn.).

Baum, Walter. Die Nullwegegruppe und ihre Verallgemeinerungen. Compositio Math. 11, 83–118 (1953).

The considerations apply to a connected simplicial complex K and in particular to its n-dimensional skeleton K^n. By a nullspheroid is meant a simplicial mapping of a triangulated n-sphere that has degree zero on every n-simplex of the image. Two nullspheroids are ν-homotopic if one can be (simplicially) deformed into the other in such a way that the intermediate maps are also nullspheroids. The ν-homotopy classes of nullspheroids, with the usual definition of addition, is the nullspheroid group Ψ^n. The kernel of the homomorphism of the nth homotopy group Π^n upon the nth homology group is denoted by Γ^n, and the subgroup of Γ^n that consists of elements of the form $\theta-\theta'$, where θ and θ' are spheroids that are free homotopic to one another, is denoted by $\Pi_0{}^n$. The kernel of the injection $\Pi^n(K^n)\to\Pi^n(K)$ is denoted by $P^n(K^n, K)$, and the subgroup of $P^n(K^n, K)$ that consists of elements of the form $\rho-\rho'$, where ρ and ρ' belong to $P^n(K^n, K)$ and are free homotopic to one another in K^n, is denoted by $P_0{}^n(K^n, K)$. It is shown that

$$\Psi^n(K)\approx\Gamma^n(K^n)/P_0{}^n(K^n, K),$$
$$\Psi_0{}^n(K)\approx\Pi_0{}^n(K^n)/P_0{}^n(K^n, K),$$

where $\Psi_0{}^n$ is a certain group related to Ψ^n very much as $\Pi_0{}^n$ is related to Γ^n. For $n=1$ and multiplication for group operation both isomorphisms reduce to the known isomorphism [Hopf, Comment. Math. Helv. 14, 257–309 (1942); these Rev. 3, 316] $G^*\approx[F, F]/[F, R]$, where F is the free group $\Pi'(K')$, and R is the kernel of the injection $\Pi'(K')\to\Pi'(K)$. The group $\Psi'(K)$ $(\approx G^*)$ is the nullpath group of the title.

To the fundamental group $G=\Pi'(K)=F/R$ Hopf associated the "homology groups" $G^2=G/[G, G]$, G^3, G^4, $\cdots$ by an algebraic algorithm. Hopf showed that G^3 is isomorphic to a subgroup $G_1{}^*$ of G^* and that $G_1{}^*\approx[F, F]\cap R/[F, R]$. The author proves that G^3 is isomorphic to $\Psi^3(K)/\Psi_0{}^3(K)$. The general theorem is that if K is aspherical in dimensions 2, 3, $\cdots$, $n-1$ then G^n is isomorphic to $\Psi^n(K)/\Psi_0{}^n(K)$.

The author sketches a proof that $\Psi^n(K)$ (and presumably also $\Psi_0{}^n(K)$) is a topological invariant of K. From this he deduces that $\Psi^n(K)/\Psi_0{}^n(K)$ is a topological invariant of K. This last is isomorphic to a group $\Delta^n(K^n)$ introduced by Hopf [ibid. 17, 307–326 (1944); these Rev. 7, 36] and not previously known to be a topological invariant of K. *R. H. Fox* (Princeton, N. J.).

Cockcroft, W. H. On two-dimensional aspherical complexes. Proc. London Math. Soc. (3) 4, 375–384 (1954).

The author obtains partial results for the problem: which subcomplexes of a two-dimensional aspherical complex are aspherical? Let L be a finite two-dimensional connected CW complex. The following is shown. (1) If $\pi_1(L)$ is free and $H_2(L)=0$, then L is aspherical. (2) If L is non-aspherical and $\pi_1(L)$ is either abelian, or a finite group, or a free group, then the complex K obtained by adjoining any collection of 2-cells to the one-skeleton of L is also non-aspherical. (3) If L is non-aspherical and has only one two-cell, then K is also non-aspherical. *J. Dugundji.*

Cohen, D. E. Spaces with weak topology. Quart. J. Math., Oxford Ser. (2) 5, 77–80 (1954).

The topology introduced by J. H. C. Whitehead for complexes is extended to general spaces with a preferred system of "test" subsets, and is called the weak topology. Specifically, closedness of a set A is equivalent to the requirement that A intersect each test set in a closed set. If the test sets are compact the space is a K space. An example of Dowker's [Amer. J. Math. 74, 555–577 (1952); these Rev. 13, 965] implies that $K\times K$ is not a K space. Some direct applications are made to CW-complexes. *D. G. Bourgin* (Rome).

Hashizume, Yoko, and Hosokawa, Fujitsugu. A note on exactness theorem. Math. Japonicae 3, 41–44 (1953).

The authors give another proof of the exactness of the cohomology sequence of a pair for the Alexander-Wallace cohomology groups [Spanier, Ann. of Math. (2) 49, 407–427 (1948); these Rev. 9, 523]. *E. Spanier.*

GEOMETRY

Schuh, Fred. A geometrical locus connected with the circumscribed circle of the pedal triangle. I, II, III, IV, V. Nederl. Akad. Wetensch. Proc. Ser. A. 57 = Indagationes Math. 16, 92–103, 129–139, 140–151, 238–249, 250–262 (1954). (Dutch. French summary)

Given a triangle ABC the author asks for the locus of a point P the pedal triangle of which with respect to ABC has a circumcircle with a given radius ρ. Using the methods of analytical geometry (rectangular coordinates), the equation of the locus is found. It is a curve C of degree six, and passes thrice through the isotropic points J and K; the tangents in J are AJ, BJ, CJ, etc.; A, B and C are double points of C. The author gives a very elaborate description of the curve, considering all kinds of special cases. Let ABC be rectangular, isosceles, equilateral; a double point of C may be a cusp; C may have other double points; it can degenerate, and so on. Several numerical examples are considered and graphs drawn.

Reviewer's remark. The author has not been aware that C is a special three-bar curve; his locus has been considered already by Alt [Z. Angew. Math. Mech. 1, 373–398 (1921)] who met it starting from a problem of kinematics. The curve C, sometimes called Alt's curve [cf. Groenman, Thesis, Delft, 1950; these Rev. 11, 621], is a three-bar sectic the three double points of which coincide with the principal foci. *O. Bottema* (Delft).

Neville, E. H. Oblique pedals. Math. Gaz. 38, 166–171 (1954).

Jeger, M. Zur Erzeugung ebener Figuren durch Projektion. Elemente der Math. 9, 101–111 (1954).

Yamashita, Chitose. An elementary and purely synthetic proof for the double six theorem of Schläfli. Tôhoku Math. J. (2) 5, 215–219 (1954).

The author deduces the double six theorem from the statement that if three skew lines all intersect three other skew lines, any transversal to the first set of three intersects any transversal to the second set [H. F. Baker, Principles of

geometry, vol. 1, Cambridge, 1922, p. 46]. He includes a useful bibliography, beginning with Schläfli's famous work [Quart. J. Pure Appl. Math. **2**, 55–65, 110–120 (1858)] and ending with "[46] This paper"! *H. S. M. Coxeter.*

Lauffer, Rudolf. Die nichtkonstruierbare Konfiguration (10_3). Math. Nachr. **11**, 303–304 (1954).

It was shown by Schroeter [Nachr. Ges. Wiss. Göttingen **1889**, 193–236] that one combinatorially possible 10_3 configuration cannot be realized in a geometry over a commutative field. The author gives coordinates for its ten points in terms of quaternions. *H. S. M. Coxeter.*

Lenz, Hanfried. Bemerkungen zur Winkelteilung. S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. **1953**, 273–281 (1954).

The author considers the use of an instrument called the angle-divider, which divides a given angle into n equal parts. He gives necessary and sufficient conditions for the possibility of constructing a segment of length x by means of a straight-edge and an angle-divider. He proves that any construction using a straight-edge and one application of compasses can be accomplished with the straight-edge and two applications of the angle-bisector (i.e., the angle divider with $n=2$). *H. S. M. Coxeter* (Toronto, Ont.).

***Baumgartner, Ludwig. Geometrie im Raum von vier Dimensionen.** Verlag von R. Oldenbourg, München-Düsseldorf, 1954. iii+112 pp. Kart. DM 6.20.

This is a text-book, in simple style, assuming only a knowledge of ordinary solid geometry, synthetic and analytic. The following extracts from the chapter headings indicate the the topics treated: Join and intersection; parallelism; orthogonality; rotation and angle; foundations of analytic geometry; general properties of polytopes; the three simplest types of polytope; Euler's formula; regular polytopes; curved spaces.

A good feature is the abundance of exercises, with answers at the end of the book. There is a useful bibliography (from Grassmann's Die lineale Ausdehungslehre . . . [Wigand, Leipzig, 1844] to Lietzmann's Anschauliche Einführung in die mehrdimensionale Geometrie [Oldenbourg, München, 1952; these Rev. **14**, 575]), but the author might well have included D. M. Y. Sommerville's "Geometry of n dimensions" [Methuen, London, 1929] and the reviewer's "Regular polytopes" [Pitman, New York, 1949; these Rev. **10**, 261]. Though aware of the pioneering work of Schläfli [Gesammelte mathematische Abhandlungen, Bd. I, Birkhäuser, Basel, 1950, p. 214; these Rev. **11**, 611], he makes no use of the Schläfli symbol $\{m, n, p\}$.

H. S. M. Coxeter (Toronto, Ont.).

Have
***Blaschke, Wilhelm. Analytische Geometrie.** 2te Aufl. Verlag Birkhäuser, Basel-Stuttgart, 1954. 190 pp. DM 19.60.

The first chapter contains many interesting details of historical background, such as the derivation of the inner and outer products of vectors from Grassmann's calculus of extension, and the application of quaternions to kinematics. The second chapter includes a discussion of Reye's configuration, formed by the centres of similitude of four spheres, which is related to Stephanos's "desmic" figure. Stereographic projection leads naturally to a discussion of inversion and linkages. The third chapter is a vectorial treatment of rotors or sliding vectors (Stäbe), couples (Stabpaare), systems of forces (Stabwerke), linear complexes, and null systems. Clifford's "dual numbers" ($a+\epsilon b$ with $\epsilon^2=0$) are used in proving the theorem of Hjelmslev (alias Petersen) and Morley: In any rectangular skew hexagon, the three common perpendiculars of pairs of opposite sides have a common perpendicular. The author remarks that the ten lines in this figure have a formal resemblance to the ten points of the Desargues configuration. There is actually a direct connection: a Desargues configuration (in the plane at infinity) is formed by the points at infinity on the ten lines and the lines at infinity in respectively perpendicular planes.

The next two chapters, on moments of inertia and quadrics, are followed by one dealing with confocal quadrics, geodesics on quadrics, Dupin's cyclides, and the tetrahedral complex formed by lines that are orthogonal to their polar lines. The final chapter is a convenient summary of the most important formulae in elementary algebra and geometry, including many that are not covered by the previous chapters; this should be useful for both teachers and students. The book is handsomely printed and has a good index.

H. S. M. Coxeter (Toronto, Ont.).

Have
***Blaschke, Wilhelm. Projektive Geometrie.** 3te Aufl. Verlag Birkhäuser, Basel-Stuttgart, 1954. 197 pp. DM 19.60.

The author has corrected a few small errors that occurred in the earlier editions [these Rev. **10**, 58], and added an extra chapter on nets (Waben). This provides a summary of several papers such as that of R. Baer [Trans. Amer. Math. Soc. **46**, 110–141 (1939); these Rev. **1**, 6]. The author draws attention to a mysterious analogy relating projective geometry to net-theory. The Pappus configuration, which exists when the field of the geometry is commutative, corresponds to the Thomsen figure, which closes when the group of the net is Abelian. Just as three applications of Pappus's theorem are needed to prove Desargues's theorem, so three applications of the closing of Thomsen's figure are needed to establish the closing of another important figure, named after Reidemeister.

The printing and binding are finer than in the first two editions, but one regrets that the portrait of von Staudt no longer appears as frontispiece. *H. S. M. Coxeter.*

Brauner, Heinrich. Kongruente Verlagerung kollinearer Räume in halbachsiale Lage. Monatsh. Math. **58**, 13–26 (1954).

In this paper the author follows up his previous investigation [Monatsh. Math. **57**, 75–87 (1953); these Rev. **15**, 56] by investigating the problem of reducing a general space collineation ϖ to a half-axial collineation $\bar{\varpi}$ (one with a single line of united points) by a congruent transformation κ, i.e. expressing ϖ as a product $\bar{\varpi}\kappa$. Four cases arise according to whether $\bar{\varpi}$ is required to be general, special, parabolic or special parabolic; and there is, naturally, a preliminary investigation of the problem as to what point ranges are congruently transformed by a general collineation ϖ. The author shows that in the four cases in question there are respectively ∞^3, ∞^2, ∞^2 and ∞^1 possible choices for κ, and he distinguishes between solutions involving direct and indirect congruence transformations. Various loci and kinematic properties of the associated families of congruent transformations are investigated. *J. G. Semple.*

Primrose, E. J. F. Real projective geometry. Math. Gaz. **38**, 185–189 (1954).

The author uses commutative involutions to define a pencil of conics without presupposing a common self-polar

triangle. However, since his methods are entirely analytic, it would be almost equally satisfactory to use the classical definition $\Sigma+\lambda\Sigma'=0$, as in the reviewer's "The real projective plane" [McGraw-Hill, New York, 1949, p. 207; these Rev. 10, 729]. There is also a section on the twisted cubic and its virtual secants or "links". When the twisted cubic is defined as the locus of the point of intersection of corresponding lines of two collineated bundles, the links arise as lines of intersection of corresponding planes of the two bundles. *H. S. M. Coxeter* (Toronto, Ont.).

André, Johannes. Über nicht-Desarguessche Ebenen mit transitiver Translationsgruppe. Math. Z. 60, 156–186 (1954).

This paper gives a group-theoretical approach to the affine minor theorem of Desargues. Define a partition of a group G (additively written) as a set of proper subgroups $U, V, \cdots$ such that every element of G is in one of the subgroups, but only zero is in more than one. A congruence K is a partition of G in which for any two groups of the partition U and V we have $G=U+V$. With these definitions it may be shown that G is abelian and that all the subgroups of the partition are isomorphic. Moreover, from G we may construct an affine plane by taking the elements of G as points, and as lines taking cosets $U+a$ with U in the congruence. Parallels are given by all cosets of the same U. The mapping $g\rightarrow g+h$, h fixed is a translation fixing the line at infinity pointwise. The translations themselves form a group isomorphic to G. Conversely, a plane with a transitive group of translations fixing the line at infinity pointwise is given by a congruence on such a group G. Equivalently, the affine minor theorem of Desargues is valid in the plane. A corresponding quasifield (or Veblen-Wedderburn system) may be regarded as the group G with operators. A quasifield is not determined uniquely by its plane, but it may contain an associative and fully distributive nucleus which is uniquely determined. *Marshall Hall, Jr.*

Leisenring, Kenneth B. A theorem in projective n-space equivalent to commutativity. Michigan Math. J. 2, 35–40 (1954).

In einer projektiven analytischen Geometrie des R_n $(n\geqq 2)$ über einem Körper wird der Satz bewiesen: sei H_0 eine Hyperebene des R_n und seien t_i $(i=1, \cdots, n+1)$ Punkte von H_0 in allgemeiner Lage; die für $k\neq m$ aus diesen Punkten durch Auslassung von t_k und t_m entstehende Restmenge sei $A_k^m=A_m^k$; durch jedes A_k^m lege man zwei von H_0 verschiedene Hyperebenen H_k^m und H_m^k; die Hyperebenen H_k^m bestimmen für $m=1, \cdots, k-1, k+1, \cdots, n+1$ einen Punkt p_k, die Hyperebenen H_m^k für $k=1, \cdots, m-1, m+1, \cdots, n+1$ einen Punkt q_m; wenn die p_k abhängig sind, so besteht zwischen den q_m eine Abhängigkeit vom selben Rang. Umgekehrt wird gezeigt, dass die Kommutativität der Multiplikation für diesen Satz notwendig ist. Für Räume ungerader Dimension wird ein analoger Schnittpunktsatz angegeben, der eine Verallgemeinerung der Möbius'schen Tetraeder-Konfiguration ist. *R. Moufang.*

Naumann, Herbert. Stufen der Begründung der ebenen affinen Geometrie. Math. Z. 60, 120–141 (1954).

The author considers seven types of configuration theorems for projective and affine planes and the corresponding algebraic systems of coordinates as given by Hilbert's coordinatization or that of the reviewer. The first three types are the classical cases of (1) the affine theorem of Pappus corresponding to commutative fields; (2) the affine theorem of Desargues corresponding to division rings; and (3) the minor theorem of Desargues corresponding to alternative division rings. The fourth type is given by a division ring with both distributive laws which is not alternative. This corresponds to the affine minor theorem of Desargues and a further special case of the minor theorem with the center of perspectivity a fixed point at infinity but the axis of perspectivity a finite line. The fifth type corresponds to the minor affine theorem alone and corresponds to a quasifield (Veblen-Wedderburn system). These have only one distributive law and include, as a special case, nearfields. Two fixed choices for the center of perspectivity on the line at infinity yield all others and hence the fifth type. This is equivalent to the dependence of the commutativity of addition in a quasifield on the remaining axioms. An example shows that the same conclusion will not hold in webs of parallels. In the sixth type both the axis and center of perspectivity are fixed. Here addition is a group but need not be abelian. In the seventh there is a further fixed line. This gives lines linear equations, but gives no properties for addition. *Marshall Hall, Jr.*

Skopec, Z. A. Certain types of plane and skew quadrilaterals in Lobačevskiĭ space. Uspehi Matem. Nauk (N.S.) 9, no. 2(60), 179–183 (1954). (Russian)

There are three types of quadrilaterals in the Lobačevskiĭ plane with proper vertices: convex, concave and self-intersecting ones (degenerate if one of the angles becomes zero or 180°). Attention is drawn to such (plane or skew) quadrilaterals for which (a) the sum of two adjacent sides is equal to the sum of the two other adjacent sides (quadrilaterals of the first kind), (b) the sum of two opposite sides is equal to the sum of the two other opposite sides (quadrilaterals of the second kind). Among the plane quadrilaterals of the first kind all three types exist, whereas among those of the second kind there are no self-intersecting ones. Two opposite vertices of a quadrilateral of the first kind at which pairs of sides with identical sum converge are called basic, the other vertices are called complementary. Necessary and sufficient condition that (a) a plane quadrilateral belong to the first kind is that the exterior bisectors of angles at basic vertices and the interior bisectors of the angles at complementary vertices belong to one pencil; (b) a plane quadrilateral belong to the second kind is that the interior bisectors belong to an elliptic pencil with center the center of the inscribed circle; (c) in a skew quadrilateral the sum of two sides be equal to the sum of two other sides is that the sides are parallel to one plane; (d) in a skew polygon of $n+1$ sides in n-dimensional Lobačevskiĭ space the sum of one group of sides be equal to the sum of the remaining sides is that the sides be parallel to one hyperplane. *D. J. Struik.*

Wald, A. Congruent imbedding in F-metric spaces. Pacific J. Math. 4, 305–315 (1954).

This article was left unpublished by the late Prof. A. Wald and probably written by him in 1934 in Vienna. It was intended to follow a paper on a similar subject by O. Taussky [Ergebn. Math. Kolloq. 6, 20–23 (1935)]. The translator, L. M. Blumenthal, has added a few comments. An F-metric space is a set S of elements (points) such that to each pair x, y is associated an element xy^2 (the squared distance) of a field F. The set of ordered n-tuples $(x_1, \cdots, x_n)$ $(x_i \in F)$ is denoted by $F_n(a_i)$ if the squared distance is given by $xy^2=\sum a_i(x_i-y_i)^2$. Under the assumption that F has

characteristic 0 it is proved that $F_n(a_i)$ has congruence order $n+3$ with respect to the class of all F-metric spaces. Let F_n denote the space $F_n(a_i)$ for $a_i=1$. Under the additional assumptions that each sum of squares of elements of F is a square and F does not contain $\sqrt{-1}$ the necessary and sufficient conditions are given in order that a $(k+1)$-tuple of an F-metric space be congruently imbeddable in F_n.
J. Haantjes (Leiden).

Convex Domains, Extremal Problems

Bambah, R. P. On polar reciprocal convex domains. Addendum. Proc. Nat. Inst. Sci. India **20**, 324–325 (1954).

In this addendum to an earlier note [same Proc. **20**, 119–120 (1954); these Rev. **15**, 607] the author establishes the best possible inequalities $27/4 \leq c(k)c(K) \leq 9$.
E. G. Strauss (Los Angeles, Calif.).

Fejes Tóth, L. Über die dichteste Horozyklenlagerung. Acta Math. Acad. Sci. Hungar. **5**, 41–44 (1954). (Russian summary)

In an earlier paper [same Acta **4**, 103–110 (1953); these Rev. **15**, 341] the author proved that every packing of equal circles in the hyperbolic plane has density less than $3/\pi$, which is the density of a special packing of horocycles. He now proves that every packing of horocycles has density $\leq 3/\pi$.
H. S. M. Coxeter (Toronto, Ont.).

Molnár, József. Kreislagerungen auf einer Kugel. Mat. Lapok **4**, 113–123 (1953). (Hungarian. Russian and German summaries)

Let $n \geq 1$ non-overlapping circles of maximum radius be drawn on a sphere. Their density D_n is defined as the sum of their areas divided by the surface area of the sphere. It is known that $D_5=1.25(2-\sqrt{2})$ and that $\lim_{n\to\infty} D_n=\pi/\sqrt{12}$. In this paper it is proved that $D_n \geq D_5$ for all $n \geq 3$. For $n \geq 1089$ this follows from an inequality due to van der Waerden. Most of the cases $n<1089$ have to be settled by calculations, the results of which for $n \leq 220$ are shown in the paper.
G. A. Dirac (London).

Hadwiger, H. Kurze Herleitung der isoperimetrischen Eigenschaft der Kugel. Elemente der Math. **9**, 97–101 (1954).

Algebraic Geometry

Turri, Tullio. Le trasformazioni birazionali cicliche del piano. Rend. Sem. Fac. Sci. Univ. Cagliari **23**, 117–143 (1953).

En utilisant la propriété des systèmes linéaires invariants par une transformation birationnelle d'avoir ses adjoints successifs invariants, et particulièrement le dernier formé de courbes rationnelles, elliptiques ou sextiques à 8 points doubles, l'auteur classe ces transformations cycliques, particulièrement celles de période supérieure à 2. En particulier, une transformation cyclique de période >2, ayant un faisceau de droites unies est ou une homologie ou birationnellement identique à une transformation quadratique; il n'existe pas de transformations birationnelles avec 4 points unis; s'il y a 5 ou 6 points fondamentaux, il n'existe pas de transformations birationnelles de période >2. S'il y a 3, 7, ou 8 points fondamentaux, il peut y avoir des transformations birationnelles de période 6, si le nombre des points fondamentaux est 3, c'est une transformation quadratique échangeant circulairement les faisceaux de droites issues des points fondamentaux; dans les autres cas (7 et 8), il n'y a pas de courbe de points unis, et la transformation est le produit d'une homographie cyclique permutable avec, ou l'involution de Geiser déterminée par les 7 points, ou celle de Bertini définie par les 8 points.
B. d'Orgeval.

Vaccaro, Michelangelo. Il discriminante di una curva del terzo ordine come determinante di una matrice cubica. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **13**, 133–156 (1954).

In analogy with the property of the determinant of the symmetric matrix of a quadratic form, the author suggests as a guiding principle that the determinant of a k-dimensional cubic matrix be defined to be a polynomial in its elements such that for a symmetric k-dimensional matrix, the vanishing of this polynomial shall be the necessary and sufficient condition that the algebraic variety defined by the associated form of order k shall have a double point. The possibility is studied in detail in the case of $3\times3\times3$ matrix.
G. B. Huff (Athens, Ga.).

Dedò, Modesto. Una dimostrazione del teorema di Lüroth. Boll. Un. Mat. Ital. (3) **9**, 141–143 (1954).

A neat argument shows that an irrational involution γ_n^1 on a line coincides with the g_n^1 defined by two of its sets.
D. B. Scott (London).

Thalberg, Olaf M. Conic involutions with a conic as coincident curve. Avh. Norske Vid. Akad. Oslo. I. **1952**, no. 1, 10 pp. (1953).

This is a sequel to an earlier paper [same Avh. **1947**, no. 1; these Rev. **9**, 464] by the same author. A conic involution is an involution of point-pairs (P, P') in the plane, such that P, P' always lie on the same conic of a given pencil. The case here considered is that in which (a) the four base points a, b, c, d of the pencil are distinct, and (b) the coincidence locus of the involution is a conic through a and b. The plane involution so defined is an obvious quadratic transform of that generated by pairs of points collinear with a fixed point and conjugate for a fixed conic; but the author does not refer to this, and works out the details directly and at length. Further papers are promised dealing with cases in which there are coincidence curves of orders $4n+2$, $4n+3$ and $4n$.
J. G. Semple (London).

Spampinato, Nicolò. Il teorema fondamentale sulle condizioni di razionalità di una superficie biduale, triduale, n-duale. Giorn. Mat. Battaglini (5) **2**(**82**), 277–295 (1954).

A classical theorem of Castelnuovo states that a surface F in complex projective space S_r is a birational image of the complex projective plane if F is a rational image of the complex plane. The author studies surfaces in the projective space S_3 over n-dual algebras [Spampinato, Rend. Accad. Sci. Fis. Mat. Napoli (4) **18**, 219–226 (1952); these Rev. **14**, 346] which are rational images of the n-dual projective plane and obtains necessary and sufficient conditions that such surfaces be rational.
G. B. Huff (Athens, Ga.).

Vesentini, Edoardo. Classi caratteristiche e varietà covarianti d'immersione. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **16**, 199–204 (1954).

This paper gives a proof that if P is a non-singular algebraic variety, of dimension p, immersed in a non-singular variety V, of dimension v, the homology classes of P dual to the characteristic classes of the normal bundle of P in V coincide with the homology classes defined by the succession of systems of equivalence on P inverse [in the sense of B. Segre, Ann. Mat. Pura Appl. (4) **35**, 1–127 (1953); these Rev. **15**, 822] to the succession of covariants of immersion of P in V. *W. V. D. Hodge.*

Boughon, Pierre. Enveloppes d'une famille à $n-1$ paramètres de variétés de dimension $n-1$ dans un espace de dimension n. C. R. Acad. Sci. Paris **239**, 23–25 (1954).

The author defines the $(n-1)$-dimensional system $\mathfrak{F}$ of $(n-1)$-dimensional cycles of an affine space A'^n with the general point $(X_1, \cdots, X_n)$ over an arbitrary field k of characteristic p, by the equation $F=0$,

$$F \epsilon k[T_1, \cdots, T_{n-1}][X_1, \cdots, X_n],$$

where $(T_1, \cdots, T_{n-1})$ is a general point of an affine $(n-1)$-dimensional space A^{n-1} over k. Let K be a perfect extension of k, and Δ an universal domain for K. The generic cycle V_t of the system is defined by $f=0$, where

$$f=F(t_1, \cdots, t_{n-1}; X_1, \cdots, X_n)$$

and the t_i, $i=1, \cdots, n-1$, are algebraically independent quantities over K. Let E be the envelope of $\mathfrak{F}$ and $Y=E\cdot V_t$. If Y is contained in the $(n-1)$-dimensional cycle w_t', which is algebraic over $k(t)$ and is such that it is not tangent to V_t in almost every point of every component of Y, and since the cycle $\sum w_t^i$, where w_t^i are all the conjugates of w_t^1 over $k(t)$, is a rational cycle over $k(t)$, then $\sum w_t^i$ is of the form $pr_{A'}W_t\cdot(P\times A')$, where P is a generic point of A^{n-1} over K. The author imposes on the W_t the following conditions: if D^n is an irreducible component of $C=W_1, \cdots, W_{n-1}$, it is verified that (A) $pr_{A'}D^n=A'^n$, (B) the coefficient of D^n in C is the unity, (C) there exist, and are transversal, the linear tangent varieties to W_i in a generic point of D^n, and (D) V_t and $pr_{A'}W_t\cdot(P\times A')$ are not tangent in almost every point of their intersection's cycle. Then, it is announced that a component M of the intersection of the cycles $V_t\cdot(P\times A')$ and C is projected on a component of E if and only if it is possible to divide the set $(1, 2, \cdots, n-1)$ into two sets G and H such that (α) the linear variety tangent to W_i in M, $i \epsilon G$, has the equation $T_i-t_i=0$, and (β) the linear variety tangent in M to W_j, $j \epsilon H$, is parallel to T_j. *P. Abellanas* (Madrid).

Roth, L. Pseudo-Abelian varieties. Proc. Cambridge Philos. Soc. **50**, 360–371 (1954).

L'autore definisce come varietà pseudo-abeliana del tipo q e di dimensione p una varietà algebrica W_p con un gruppo continuo permutabile $G\ \infty^q$ $(1\leqq q\leqq p-1)$ di automorfismi. Le traiettorie V_q di G sono varietà di Picard (che l'autore suppone a moduli generali) birazionalmente equivalenti costituenti una congruenza $\{V_q\}\ \infty^{p-q}$. Una W_p contiene una seconda congruenza $\{V_{p-q}\}$, birazionalmente equivalente ad una V_q di Picard, costituita da $\infty^q\ V_{p-q}$ esse stesse birazionalmente equivalenti, e seganti sulla generica V_q una involuzione i_d il cui ordine $d=[V_q, V_{p-q}]\geqq 1$ (determinante di W_p) è un importante carattere di W_p, analogo al divisore di una V_p di Picard. Se $d=1$, $\{V_q\}$ e $\{V_{p-q}\}$ sono birazionalmente equivalenti a V_{p-q} e V_q rispettivamente e si può rappresentare W_p sul prodotto $V_q\times V_{p-q}$, mentre se $d>1$ si può rappresentare W_p su una varietà d-pla W_p^* di determinante unitario. Da tale rappresentazione l'autore deduce: (1) un'equivalenza per il sistema canonico $|X_{p-1}|$ di W_p; (2) la limitazione $q_2\geqq q_2^*+q$ essendo q_2 l'irregolarità superficiale di W_p e q_2^* l'irregolarità della congruenza $\{V_q\}$; (3) una rappresentazione analitica di W_p nel caso in cui i_d sia ciclica. I sistemi canonici $\{X_k\}$ di W_p, come già $|X_{p-1}|$ sono generati da traiettorie oppure hanno ordine zero, e quindi gli invarianti canonici di W_p (e cioè i numeri di intersezione $[X_{i_1}\cdots X_{i_r}]$ con $i_1+\cdots+i_r=(r-1)p$) sono tutti nulli. Lefschetz ha provato che il genere aritmetico P_a di una V_p di Picard (definito medianta la formula di postulazione) è $(-1)^{p-1}$: lo stesso si ha per una W_p di determinante unitario; e ciò vale pure per $d>1$ purchè si supponga che P_a sia un carattere numerativo di W_p. Hanno pure interesse alcuni lemmi e risultati preliminari ottenuti dall'autore: egli prova tra l'altro che ogni V_p di irregolarità superficiale p con una ipersuperficie canonica pura d'ordine zero, che non contenga una congruenza d'irregolarità p, è una varietà di Picard. *D. Gallarati* (Genova).

Differential Geometry

Mirguet, J. Remarques sur les équivalences géométriques de la dérivabilité seconde des orthosurfaces. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **40**, 608–612 (1954).

$f(x, y)$: fonction définie sur un ensemble ouvert G du plan des (x, y), pourvue de dérivées premières (finies) continues par rapport à (x, y). S (orthosurface): surface représentative en axes rectangulaires de $z=f(x, y)$. (x_0, y_0): point de G, $A=(x_0, y_0, f(x_0, y_0))$. G. Bouligand dans son "Introduction à la géométrie infinitésimale directe" [Vuibert, Paris, 1932, §147] définit une notion de dérivabilité seconde pour f en (x_0, y_0) en demandant pour le contingent de courbure normale de S en A de contenir dans tout plan normal un cercle et un seul de rayon borné inférieurement par un nombre positif. Cette hypothèse de dérivabilité seconde implique la validité des théorèmes de Meusnier et d'Euler pour S en A. Dans la présente note l'auteur définit une notion de dérivabilité seconde qu'il appelle "précisée", faisant intervenir deux points mobiles M' et M'' sur S voisins de A et il montre qu'elle implique la dérivabilité seconde selon G. Bouligand. Nous avons là l'analogue dans le cas à deux dimensions de la courbure de Alt pour une courbe plane dont l'existence implique celle de la courbure tangente [cf. van der Waag, Nederl. Akad. Wetensch. Proc. Ser. A. **55**=Indagationes Math. **14**, 92–103, 275–286 (1952); ces Rev. **14**, 83]. *Chr. Pauc* (Nantes).

Mineo, Corradino. Superficie delle quali una semplice infinità di geodetiche sono eliche su cilindri ortogonali a una direzione fissa. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **16**, 165–170 (1954).

The author has already studied these surfaces [cf. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) **3**, 175–182 (1942); these Rev. **8**, 225]. The surface S is referred to a geographic system of coordinates with respect to a fixed direction r, and the author sets himself again the problem of determining those surfaces with the property that a family of lines of constant longitude shall also be a family of geodesics. The equation which expresses this condition and the two Codazzi-Mainardi equations are solved for the coefficients of the second fundamental form, there being three cases. Then relying on a paper of the author on sur-

faces referred to geographic coordinates [Giorn. Mat. Battaglini (3) 1(48), 185–229 (1910)], the equations of the surfaces S are determined except for quadratures from their second and third fundamental forms. *A. Schwartz.*

Gentile, Maria Luisa. Una formula sull'incidenza di piani infinitamente vicini, con applicazione alle linee principali di una superficie. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. 87, 43–50 (1953).

A. Terracini has studied the conditions under which a system of ∞^1 planes of S_6, all tangent to a curve, will be such that two consecutive planes intersect with order of approximation θ [Scritti matematici offerti a Luigi Berzolari, Pavia, 1936, pp. 449–478]. The present author continues this work and without assuming special analytic representations, gets a necessary and sufficient condition for an intersection with order of approximation $\theta \geqq 8$. Terracini applied his results to the study of a system of planes tangent to the principal lines of a surface in S_6 [Boll. Un. Mat. Ital. (3) 7, 247–252 (1952); these Rev. 14, 498]. Here we have the following extension: If for a surface S of S_6, a system of planes tangent to a non-planar system of principal lines is such that the intersection of two consecutive planes is of the order of approximation $\geqq 8$ and if the system of principal lines in question is of multiplicity at least 4, then the order of multiplicity must be 5.

A. Schwartz (New York, N. Y.).

Corio, Arnaldo. Sopra una notevole famiglia di supergeodetiche. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. 87, 99–111 (1953).

In the first part the author considers upon a surface Σ a particular family of supergeodesics related to the consideration of the plane sections through a point of Σ having, at that point, an osculating circle with 4-point contact. The supergeodesics so obtained coincide with the curves of the same name treated by D. B. Dekker [Pacific J. Math. 1, 53–57 (1951); these Rev. 13, 383]. The differential equations of the supergeodesics are then integrated for the case of the cone, the torus, the paraboloid and the ellipsoid. In the latter part of the note he studies the plane sections through a point of Σ having an osculating circle with 5-point contact at that point. He concludes that through an arbitrary point P of a surface Σ there pass 12 planes whose sections with Σ have, in P, an osculating circle with 5-point contact.

E. T. Davies (Southampton).

Vaccaro, Giuseppe. Cerchi iperosculatori ad una superficie in un punto e questioni connesse. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 16, 35–41 (1954).

The author points out that the number of plane sections at a point P of a surface in S_3 having an osculating circle with 5-point contact in P is 10 and not 12 as stated by Corio in the note reviewed above.

The author first finds the number of 4-point tangents to an algebraic F^n in points of a line C on which there are no singular points, extending the result to the case where F^n possesses on C double points of simple type. He then shows that the problem studied by Corio can be reduced to that of the determination of the number of 4-point tangents to an F^n in points of a line on which are situated two $(n-1)$-ple points of F^n. He also shows that it can be reduced to the problem of determining the planes issuing from a point P of a surface Φ belonging to a hypersphere in S_4 and having in P 5-point intersection with Φ. *E. T. Davies.*

***Sbrana, Francesco. Su alcune proprietà delle curve sghembe.** Convegno Internazionale di Geometria Differenziale, Italia, 1953, pp. 263–265. Edizioni Cremonese, Roma, 1954. 4000 Lire. on order

Let t, n, b be the tangent, principal normal, and binormal, and let c and T be the first and second curvatures of a curve L. Let a unit vector w be chosen in the osculating plane such that dw/ds has the direction of b. Then $c=d\phi/ds$, where ϕ is the angle w makes with n. Let N be a unit vector in the normal plane such that dN/ds has the direction of t. Then $T=d\theta/ds$, where θ is the angle N makes with n. Let v be a unit vector in the rectifying plane such that dv/ds has the direction of n. Then $d\psi/ds=0$, where ψ is the angle v makes with b. It is also pointed out that one can arrive at these facts by considering L as a curve on a surface which admits it as an asymptotic curve, as a curve on a surface which admits it as a line of curvature, and as a curve on a surface which admits it as a geodesic. *A. Schwartz.*

Gołąb, S. Sur quelques propriétés des courbes planes. Ann. Polon. Math. 1, 91–106 (1954).

Let $\widehat{AB}$ be an arc of a parabola and $\overline{AB}$ the corresponding chord. If p denotes the area bounded by $\widehat{AB}$ and $\overline{AB}$ and if P denotes the area of the circumscribing rectangle having $\overline{AB}$ as one side, then it has been known since Archimedes that $p/P=2/3$. For a general plane curve the ratio of the corresponding magnitudes p/P is not equal to $2/3$. This paper gives a series of propositions concerning the limiting behavior of the ratio p/P as the arc length $\widehat{AB}$ approaches zero. In the general case the limit is $2/3$, but if the arc contains points of zero or infinite curvature, or if the arc is not of sufficient differentiability the resulting conclusions must be modified. *S. B. Jackson* (College Park, Md.).

Gołąb, S. Les courbures (ordinaires) d'une courbe située sur une hypersurface et les courbures géodésiques et normales ainsi que la torsion géodésique de cette courbe. Ann. Polon. Math. 1, 81–88 (1954).

For a curve on $V_{n-1} \subset V_n$ the author [Ann. Soc. Polon. Math. 22, 97–156 (1950); these Rev. 11, 690] has defined quantities $\alpha_2, \cdots, \alpha_{n-1}, \gamma, \beta_2, \cdots, \beta_{n-1}$ in terms of the principal curvatures of the curve. Put $\alpha=(\sum\alpha^2)^{1/2}$, $\beta=(\sum\beta^2)^{1/2}$; α is called the geodesic curvature of the curve, β the geodesic torsion, and γ the normal curvature. If $\alpha\equiv 0$ the curve is called a geodesic; if $\beta\equiv 0$ it is a line of curvature; and if $\gamma\equiv 0$ it is an asymptotic line. Then it is proved that for $n\geqq 4$, if a geodesic is plane, it is a line of curvature; and that if a line of curvature is plane, the ratio γ/α is a constant. Moreover, if a curve is asymptotic of the second order it is asymptotic of arbitrary order $(\leqq n-1)$.

C. B. Allendoerfer (Seattle, Wash.).

Strubecker, Karl. Über Potentialflächen. Arch. Math. 5, 32–38 (1954).

Es handelt sich um die geometrische Deutung der analytischen Integralflächen der logarithmischen Potentialgleichung $\Delta z=\partial^2 z/\partial x^2+\partial^2 z/\partial y^2$ als Minimalflächen des isotropen Raumes vom Bogenelement $dx^2+dy^2+0\cdot dz^2$ bzw. um die der Lösungen des Dirichletschen Problems

$$O^*=\frac{1}{2}\int\int_{(\mathfrak{B})} \nabla z \, dxdy=\min.$$

Das kommt darauf hinaus O^* als Relativoberfläche und die isotrope Einheitskugel $z=\frac{1}{2}(x^2+y^2)$ als Eichfläche aufzu-

fassen. Diese Deutung beruht gruppentheoretisch darauf, dass (zwar nicht der Relativoberfläche eines Flächenstückes, wohl aber) den Extremalen des Variationsproblems $\delta O^*=0$, welche, gemäss $H=\frac{1}{2}\Delta z=0$, durch das Verschwinden der isotropen mittleren Krümmung invariant definiert sind, invariante Bedeutung gegenüber den Bewegungen des isotropen Raumes zukommt. Man wird also die Theorie der euklidischen Minimalflächen mit der der isotropen Minimalflächen (Potentialflächen $z=\Re f(x+iy)$) zu vergleichen versuchen und Analogien erwarten, insbesondere dann, wenn den euklidischen Bewegungen und Symmetrien isotrope Bewegungen und isotrope Symmetrien gegenübertreten. Dies belegt Verf. mit zwei bekannten Sätzen von H. A. Schwartz: (1) enthält die reelle Potentialfläche Φ eine reelle eigentliche nicht isotrope gerade Linie, so ist diese Gerade (im isotropen Sinne) eine Symmetrieachse der Potentialfläche Φ; (2) schneidet die reelle Potentialfläche Φ die reelle isotrope Ebene π in einer (geodätischen) isotropen Krümmungslinie, so ist π (im isotropen Sinne) eine Symmetrieebene der Potentialfläche. Zum Beweis löst Verf. das isotrope Björlingsche Problem, durch einen reellen analytischen Anfangsstreifen eine isotrope Minimalfläche, d.h. Potentialfläche zu legen. Dabei ergibt sich noch als weiteres Resultat: besitzt eine Fläche des isotropen Raumes einen geodätischen Krümmungsstreifen, so liegt dessen Streifenkurve in einer isotropen Ebene π und der Streifen ist zylindrisch und (im isotropen Sinne) zu π normal. Umgekehrt ist jeder isotrope zylindrische Krümmungsstreifen auch geodätisch. *M. Pinl* (Köln).

Hartman, Philip, and Wintner, Aurel. Umbilical points and *W*-surfaces. Amer. J. Math. **76**, 502–508 (1954).

Let S be a (small piece of a) special W-surface of class C^3, that is, a Weingarten surface for which one principal curvature is a monotone decreasing function of the other in a neighborhood of an umbilic. It is proved that, unless S is a part of a plane or of a sphere, the umbilics, if any, are isolated and their indices are negative. As a consequence it follows that if a closed orientable surface of genus zero is a special W-surface of class C^3, then S is a sphere. This improves a theorem of H. Hopf, by removing the analyticity assumptions. The proof makes use of the early work of the authors on the asymptotic behavior of solutions of nonlinear elliptic partial differential equations in two variables [same J. **75**, 298–334 (1953); these Rev. **14**, 1119]. *S. Chern* (Chicago, Ill.).

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Finikov, S. P. Two problems of contemporary differential geometry. Vestnik Moskov. Univ. Ser. Fiz.-Mat. Estest. Nauk ~~1953~~, 3–14 (1953). (Russian)

This is a report on recent work done, mainly by Moscow geometers, in two fields which are essentially in the ninteenth century tradition of Bianchi-Darboux. The first field is that of conjugate sets and congruences, in relation to Laplace transformations. Here considerable attention is paid to the work of T. Koz'mina [C. R. (Doklady) Acad. Sci. URSS (N.S.) **55**, 183–185 (1947); these Rev. **8**, 531], which was independently supported by S. S. Chern [Proc. Nat. Acad. Sci. U. S. A. **30**, 95–97 (1944); Sci. Rep. Nat. Tsing Hua Univ. **4**, 328–336 (1947); these Rev. **5**, 217; **10**, 65]. Other papers on this subject, discussed by the author, include those of R. V. Smirnov [Doklady Akad. Nauk SSSR (N.S.) **71**, 437–439 (1950); these Rev. **11**, 616], and T. A. Šul'man [ibid. **85**, 501–504 (1952); these Rev. **14**, 316]. The second field for which a survey is given is that of so-called stratified pairs of congruences (the name is due to Fubini), a topic intimately related to the first. The author, who reopened this domain in 1927, reports mainly on his own work [see, e.g., Mat. Sbornik N.S. **29(71)**, 349–370 (1951); these Rev. **13**, 773]; and also gives an account of thesis work done by several students at Moscow University on Laplace transformations as well as on stratified pairs. *D. J. Struik* (Cambridge, Mass.).

Finikov, S. P. On the problem of stratification of a pair of complexes. Uspehi Matem. Nauk (N.S.) **9**, no. 1(59), 125–130 (1954). (Russian)

The concept of a stratified pair of congruences (for which a family of surfaces is necessary) is generalized to complexes. Two complexes are called stratified if a family of curves is given, and at the points of intersection with the rays of one complex osculating planes to the curves pass through corresponding rays of the other complex, and conversely. This is a natural generalization, since the property holds for stratified pairs of congruences, taking the asymptotic lines of the family of surfaces into account. The analytical apparatus to deal with these complexes is set up by means of a tetrahedron of reference, of which two vertices A_1, A_2 lie on a ray of one complex, and the two other vertices A_3, A_4 on the corresponding ray of the other complex. Then the infinitesimal projective displacement $dA_i=\omega_i^k A_k$ $(i, k=1, 2, 3, 4)$ can be normalized by $\omega_2^3=\omega_1^4$; the families of curves can be taken to satisfy the equations $\omega_1^4=\mu\omega_1^3$, $\omega_2^4=\nu\omega_1^3$; $\omega_1^4=\mu'\omega_1^3$, $\omega_2^4=\nu'\omega_1^3$. Special attention is paid to the case $\omega_2^3=\omega_1^4$, $\omega_3^2=\omega_4^1$, in which the two complexes [12] and [34] have a common tangential complex. *D. J. Struik* (Cambridge, Mass.).

Laktanova, N. V. A stratifiable pair of surfaces. Doklady Akad. Nauk SSSR (N.S.) **92**, 473–474 (1953). (Russian)

A pair of surfaces is called stratified if between the tangent planes a mutual one-to-one correspondence exists with the following property: there exists a three-parametric system S of congruences W, the similar foci of which lie, correspondingly, in the tangent planes to the first or second surface. It is shown how this concept is connected with that of the stratification of manifolds of plane elements. Two such manifolds of four dimensions can be obtained by establishing a mutual one-to-one correspondence between the points of two surfaces A_1 and A_2 and then defining a correlation between the points M_1 (corr. M_2) and the lines l_1 (corr. l_2) in every pair of corresponding tangent planes; the centers M_1 (M_2) of the elements in the tangent planes to A_1 (A_2) are then connected with the correlated lines l_1 (l_2) of the other surface. It is also shown how this concept of a stratified pair of surfaces is a generalization of the concept of a stratified pair of congruences. Some properties of these pairs of surfaces are announced. *D. J. Struik.*

Rybakov, V. N. Binormal families of congruences. Doklady Akad. Nauk SSSR (N.S.) **93**, 13–14 (1953). (Russian)

A ruled surface is called binormal if its generators are the binormals of its line of striction. A congruence of lines can be decomposed into a family of such binormal surfaces. This paper enumerates a number of properties of such families. *D. J. Struik* (Cambridge, Mass.).

Backes, F. Sur un couple de cercles engendrant des congruences doublement stratifiables. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **40**, 613–620 (1954).

Ce travail fournit une nouvelle application de la méthode du pentasphère mobile oblique, développée par l'auteur dans son mémoire, Acad. Roy. Belgique. Cl. Sci. Mém. Coll. in 8°. (2) **26**, no. 1613 (1951) [ces Rev. **13**, 686]. Il s'agit d'une généralisation des couples de congruences rectilignes doublement stratifiables, les congruences de droites étant remplacées par des congruences de cercles (C), (C'), les cercles générateurs dépendant de deux paramètres u, v. La question est d'établir l'existence de couples (C), (C') dont les cercles générateurs possèdent chacun des sphères focales les périsphères se correspondant, et qui soient tels que, par chacun des cercles C de l'une quelconque des deux congruences, on puisse mener une infinité simple de sphères, variant avec C de telle sorte que leurs points caractéristiques soient sur le cercle C' correspondant de l'autre congruence. Cette existence est prouvée à partir de résultats obtenus dans le mémoire cité. *P. Vincensini.*

Kovancov, N. I. On the projective theory of a complex of lines. Doklady Akad. Nauk SSSR (N.S.) **95**, 917–920 (1954). (Russian)

When a line complex is referred to a moving trihedral $A_1A_2A_3A_4$ an infinitesimal displacement is given by $dA_i=\omega_{ik}A_k$ ($i, k=1, 2, 3, 4$). If A_1 and A_2 are chosen on a ray of the complex and $A_1A_2A_3$, $A_1A_2A_4$ are coordinate planes, then $\omega_{13}+\omega_{24}=0$. Exterior differentiation of this relation leads to three linear equations among ω_{ik} involving six parameters $x_1, x_2, \cdots, x_6$. Any ruled surface of the complex is given by $\omega_{23}=p\omega_{24}$, $\omega_{14}=q\omega_{24}$ and on each ruling there are two contact points determined by the roots of $pt^2-2t-q=0$. If the roots are equal ($pq=-1$), the points coincide and the surface is developable. The main part of this note deals with the principal surfaces of the line complex. Analogously to the theory of quadric surfaces, the characteristic equation is

$$\begin{vmatrix} x_1 & x_2 & x_3-s \\ x_2 & x_4-s & x_5 \\ x_3-s & x_5 & x_6 \end{vmatrix}=0.$$

If the three roots are distinct the complex has three characteristic surfaces. If two of the roots are equal the principal surface is indeterminate; it is a complex of "projective revolution". Finally if all three roots are equal the complex is linear. *M. S. Knebelman* (Pullman, Wash.).

Grincevičus, K. I. The complex of lines in affine space. Doklady Akad. Nauk SSSR (N.S.) **92**, 695–698 (1953). (Russian)

The method of the canonical moving trihedron, which Finikov has applied to the study of congruences, is here applied to complexes of lines in affine three-space. The conditions on the ω^i, ω_i^k in the equations $dA=\omega^ie_i$, $de_i=\omega_i^ke_k$, $h, i, l=1, 2, 3$ are written out; e_3 lies along the complex line l; ω_3^1, ω_3^2, ω^1 can be independently selected. Applications are made for geometrical concepts belonging to neighborhoods of the first, second and third order. Reference is made to congruences studied by P. Mentré [Thesis, Paris, 1923]. *D. J. Struik* (Cambridge, Mass.).

Tuganov, N. G. On the indicatrix of a surface. Doklady Akad. Nauk SSSR (N.S.) **94**, 189–192 (1954). (Russian)

The author has studied before the congruence of indicatrices of a surface in ordinary differential geometry [same Doklady (N.S.) **88**, 217–220 (1953); these Rev. **14**, 792]. In the present paper he continues his study for the case of surfaces in affine space. As the indicatrix he takes a second-order curve in the tangent plane whose asymptotes coincide with the tangent asymptotes of the surface. Taking its equation with respect to the trihedron of the surface as $xy=m$, $z=0$ (m the parameter of the indicatrix), the foci of the congruence are given by equations of the form $xy=m$, $z=0$, $x\omega^2+y\omega^1=0$, $ux^2\omega^1+uy^2\omega^2+dm=0$ [notation in accordance with S. P. Finikov's book, Cartan's method of exterior forms in differential geometry, OGIZ, Moscow-Leningrad, 1948; these Rev. **11**, 597]. Among the properties investigated are the envelope, the line of centers, and the quadric passing through two adjacent indicatrices. *D. J. Struik* (Cambridge, Mass.).

Gasapina, Umberto. Su una proprietà metrica delle flessioni delle superficie sviluppabili. Boll. Un. Mat. Ital. (3) **9**, 160–163 (1954).

Let (S, R) be a congruence of spheres $\theta(u, v)$ having centers $C(u, v)$ on a surface S and radii $R(u, v)$. A deformation of the congruence is obtained by deforming the surface S (with preservation of the first fundamental form) and keeping $R(u, v)$ unchanged. If S is a developable surface and if all the spheres θ pass through a fixed point, then (and only then) can (S, R) be deformed continuously into congruences (S_a, R), each sphere of (S_a, R) being orthogonal to a fixed sphere of radius a, a changing with the deformation. Moreover, with one exception, the congruences (S, R), with S developable but not a plane and the spheres passing through a point, have the characteristic property that if S is rolled out onto a plane the two folds of the spheres' envelope become two distinct lines. The exception is the case where S is a cone and all the spheres pass through its vertex. In this case the two lines contract into one point of the plane. *A. Schwartz* (New York, N. Y.).

Šulikovskiĭ, V. I. An invariant criterion for a Liouville surface. Doklady Akad. Nauk SSSR (N.S.) **94**, 29–32 (1954). (Russian)

The search for a quadratic integral of the geodesics of a surface with $ds^2=g_{ik}du^idu^k$ has led V. V. Vagner [Trudy Sem. Vektor. Tenzor. Analizu **5**, 246–249 (1941); these Rev. **8**, 602] to a system of partial differential equations of the third order on which the problem depends. In the present paper the integrability conditions of this problem are written out and conditions given under which there will be four, three, two and one quadratic integral. These conditions involve the rank of a certain system of equations of which the coefficients depend in a rather complicated way on the total curvature of the surface and its first and second derivatives. The relation of this problem to Liouville surfaces has already been pointed out by Darboux [Leçons sur la théorie générale des surfaces, t. III, Gauthier-Villars, Paris, 1894, Ch. 2]. *D. J. Struik* (Cambridge, Mass.).

Ryžkov, V. V. On a transformation of a pair of imposed surfaces. Doklady Akad. Nauk SSSR (N.S.) **95**, 25–27 (1954). (Russian)

Two euclidean spaces R_N' and R_N'', imposed on a euclidean space R_N are given by the radius vectors $\overrightarrow{OM'}=\bar{x}$ and $\overrightarrow{OM''}=\bar{y}$, where O is a fixed point in R_N. The pair of points x, y is carried into the pair $\bar{\xi}$, $\bar{\eta}$ by means of the transformation $\bar{\xi}=k\bar{x}$, $\bar{\eta}=k\bar{y}$, which is a similitude when $k=$constant. The case studied in this paper is that in which

$k=c/(\bar{x}^2-\bar{y}^2)$, c a constant, these transformation are called quasisimilar. The behavior of certain pairs of n-dimensional surfaces is studied, especially under Laplace transformations.
D. J. Struik (Cambridge, Mass.).

Bazylev, V. T. Quasi-Laplacian transformations of p-surfaces of a space P_n. Doklady Akad. Nauk SSSR (N.S.) **92**, 453–455 (1953). (Russian)

R. V. Smirnov [same Doklady (N.S.) **71**, 437–439 (1950); these Rev. **11**, 616] investigated transformations of Laplace of p-conjugate systems of projective n-space P_n $(n\geqq p)$. In the present paper are studied transformations, analogous to these Laplace transformations, but adapted to p-dimensional surfaces S_p which do not have p-conjugate systems. Special attention is paid to S_2 in P_6, and to S_p in P_{p+2} in general. For $n>p+2$ one of the results is a generalization of a theorem by C. Segre [Atti Accad. Sci. Torino **42**, 559–591 (1907)]. *D. J. Struik* (Cambridge, Mass.).

Dubnov, Ya. S. A propos of Peterson's equations. Uspehi Matem. Nauk (N.S.) **9**, no. 1(59), 101–106 (1954). (Russian)

The recent publication of K. M. Peterson's dissertation "On the bending of surfaces" [Istor.-Mat. Issled. **5**, 87–112 (1952)] gives the author the opportunity to return to some observations he made earlier [Trudy Sem. Vektor. Tenzor. Analizu **6**, 17 (1941)]. Peterson (1828–1881), in his work on differential geometry from 1853 onward, anticipated in several respects results commonly associated with the names of Bonnet and Codazzi. Led by some of Peterson's ideas, the present author shows how the equations of Codazzi can be derived from the fact that the mapping of a line of curvature on a surface on its spherical representation is a Combescure transformation, and hence keeps $k_g ds$ unchanged, k_g being the geodesic curvature. This method, which also uses Rodrigues' formula, leads to the Codazzi equations written in terms of the lines of curvature as parametric lines. In this paper we also find a discussion of the question: Given two functions $E(u,v)$, $G(u,v)$, to find all surfaces for which $ds^2=Edu^2+Gdv^2$, the lines $u=$const., $v=$const. being lines of curvature. When e, g, the coefficients of the third fundamental form, are unequal, this question leads to a differential equation in E, G of the fifth order. When one of the e, g is zero, we are led to the surfaces of Monge with $ds^2=du^2+(U+V)^2dv^2$, $U=U(u)$, $V=V(v)$; in this case $e=U''$: $(c-U'^2)$, $g=c-U'^2$. *D. J. Struik.*

Yanenko, N. N. On the theory of the imbedding of surfaces in a multi-dimensional Euclidean space. Trudy Moskov. Mat. Obšč. **3**, 89–180 (1954). (Russian)

This paper presents a complete theory of imbedding of a space V_m in a euclidean space E_{m+q}. It is primarily concerned with the questions of deformability of V_m and with the role played by the three numerical invariants, rank, class and type, in this theory. A distinction is made between two kinds of deformation: if the deformation of $V_m\subset E_{m+q}$ into $\bar{V}_m\subset E_{m+q}$ is induced by a deformation of V_{m+s} into $\bar{V}_{m+s}$, both contained in E_{m+q}, the deformation is called a codeformation. If there does not exist such a $V_{m+s}\infty\bar{V}_{m+s}$, the deformation is called proper and it is this kind of deformation that is discussed. The first four chapters are devoted to algebraic preliminaries while the last six chapters are used to obtain the geometric results. It is quite detailed even though most of the results have been previously given [Doklady Akad. Nauk SSSR (N.S.) **64**, 641–644; **65**, 449–452 (1949); these Rev. **11**, 395, 396]. The additional results can be stated as: the class of $V_m\subset E_{m+q}$ admitting infinitesimal deformations is projectively invariant. There are other projective theorems, but they are rather obvious consequences of the fact that the type of V_m is a projective invariant. *M. S. Knebelman* (Pullman, Wash.).

Green, L. W. Surfaces without conjugate points. Trans. Amer. Math. Soc. **76**, 529–546 (1954).

The author, after a preliminary study of global properties of solutions of the Jacobi equation, is able to deduce divergence properties of geodesics on certain surfaces. The surfaces considered are all complete, Riemannian manifolds of class C^3 with curvature bounded below. Such a surface is called a manifold with pole P if it is simply-connected and no geodesic through P contains a point conjugate to P; if in addition it has property (I) no conjugate points, i.e., shortest paths are unique, or (II) no conjugate or focal points, i.e., perpendiculars from point to line are unique, then it is denoted M(I) or M(II) respectively. Two geodesics are said to diverge if one of them contains a sequence of points from which the distances to the other are an unbounded sequence.

The following are among the principal results. (1) Two geodesic rays from the pole P must diverge if either (a) P has no focal points on any ray with initial point P or (b) P is interior to a region of poles. Corollary. On an M(I) two geodesic rays with the same initial point diverge. (2) If an M(II) contains two non-divergent (hence non-intersecting) geodesics the Gaussian curvature at any point between them is zero. Corollary. If an M(II) is analytic it is either flat or all geodesics diverge.

Using these and further results the author is able to improve substantially certain theorems of Morse and Hedlund [Trans. Amer. Math. Soc. **51**, 362–386 (1942); these Rev. **3**, 309] by removing the assumption of Poisson stability. For example, he shows that if there are no conjugate points the geodesic flow is transitive in the phase space of an orientable (non-orientable) surface of genus greater than one (greater than two). *W. M. Boothby* (Evanston, Ill.).

Bompiani, Enrico. Topologia differenziale. VI. Invarianti topologici di elementi di una calotta. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **15**, 242–248 (1953).

[For parts I–V see these Rev. **11**, 689; **12**, 283.] A cap is defined to be the set of quadratic curvilinear elements E_2 of a surface passing through a fixed point (its center). A particular E_2 at (0, 0) is given by $y=\lambda x+\mu x^2+$terms of higher order. It is shown that the study of the point transformations which leave four E_2 invariant is equivalent to the study of the homotheties of given center of an affine plane. The set of E_2 such that no two of them belong to the ∞^1 system of E_2 determined by four others admits a single invariant of the second order. The group of point transformations which leaves three E_2 invariant is equivalent to the group of quadratic transformations of the projective plane having two fundamental points fixed and a third on a line through the center of the cap. The point transformations which leave a pencil of E_2 invariant can be considered to be the quadratic transformations of a projective plane with two common fundamental points which reduce to the identity in the first-order neighborhood of the center.
C. B. Allendoerfer (Seattle, Wash.).

Dumitraş, V. Sur les espaces A_3 qui admettent une rotation. Acad. Repub. Pop. Române. Stud. Cerc. Mat. 4, 213–232 (1953). (Romanian. Russian and French summaries)

Affine spaces A_2 and A_3 with groups of automorphisms have been studied by G. Vranceanu [Acad. Repub. Pop. Române. Bul. Şti. Ser. Mat. Fiz. Chim. 1, 813–821 (1949); these Rev. 14, 1123] and St. Petrescu [ibid. 2, 639–646 (1950); these Rev. 14, 1123; Acad. Repub. Pop. Române. Stud. Cerc. Mat. 2, 322–358 (1951)]. The present paper deals with the special case that the A_3 admits a rotation at every point, defined by transformations of congruences of the form $d\bar{s}^1=\cos\theta\, ds^1-\sin\theta\, ds^2$, $d\bar{s}^2=\sin\theta\, ds^1+\cos\theta\, ds^2$, $d\bar{s}^3=ds^3$, which conserve the forms $r_{cd}ds^c\delta s^d$, $\gamma_{bd}ds^b ds^d$, where r_{cd}, γ_{bd} are the two contracted curvature tensors $\gamma^a{}_{bad}=r_{bd}$, $\gamma^a{}_{cad}=r_{cd}$. These tensors are taken in the canonical form,

$$r_{12}=M,\quad r_{23}=r_{31}=0,\quad \gamma_{11}+\gamma_{22}+\gamma_{33}=1,\quad \gamma_{ik}=-\gamma_{ki},\quad i,k=1,2,3,\quad i\neq k.$$

It is found that for these spaces the coefficients of connection can be made to satisfy the relations

$$\gamma^1{}_{31}=\gamma^2{}_{32},\quad \gamma^2{}_{31}=-\gamma^1{}_{32},\quad \gamma^3{}_{11}=\gamma^3{}_{22},\quad \gamma^3{}_{12}=-\gamma^3{}_{21},$$
$$\gamma^2{}_{11}=-\gamma^1{}_{21}=m,\quad \gamma^2{}_{12}=-\gamma^1{}_{22}=n,\quad \gamma^1{}_{13}=\gamma^2{}_{23},$$

all others zero except $\gamma^3{}_{33}$. The further classification depends on the invariant $A=\partial n/\partial s^1-\partial m/\partial s^2-m^2-n^2$. For $A=0$ the coefficients m and n can be made to vanish, the rotation will be $\theta=$const. and all the coefficients of connection are constants. For the case $A\neq 0$ we refer to the paper.

D. J. Struik (Cambridge, Mass.).

Čech, Eduard. Géométrie projective différentielle des correspondances entre deux espaces. I, II, III, IV, V, VI, VII. Čehoslovack. Mat. Ž. 2(77), 91–107, 109–123, 125–148, 149–166, 167–188 (1952); 297–331 (1953); 3(78), 123–137 (1953). (Russian. French summaries)

The first three parts of this sequence were previously published in French in Časopis Pěst. Mat. Fys. 74, 32–48; 75, 123–136, 137–158 (1950); these Rev. 12, 534; 13, 158. The first section of part IV gives the complete solution of the problem of obtaining all transformations which carry a congruence L in S_n into L' of S_n' and such that a ruled surface R of L is carried into a ruled surface L' of S_n', asymptotic lines of R being carried into those of R'. The second section deals with correspondences between rectilinear congruences in S_3 and S_3' for which there exist tangential collineations which carry a line p through a point A into a line q through the same point and which depends only on the plane pq. This problem is solved completely only in the case when L has two distinct families of developable surfaces. Part V deals with the existence of ∞^s collineations between projective spaces S_n and S_n' $(s<n)$. The problem is solved for $s=1$ and for $n=3$, $s=2$. Part VI is devoted to defining K-principal lines and K-principal hyperplanes for a pair of projective spaces and the problem of projective correspondences admitting K-principal hypersurfaces is solved. The results are then applied to the problem of projective deformations of hypersurfaces, part VII dealing with non-singular deformations, the main theorem being that a parabolic layer of hypersurfaces of S_n admits non-singular projective deformations which depend $2n+2$ functions of one parameter.

M. S. Knebelman.

Iwahori, Nagayoshi. On an orthogonal invariant of two linear spaces. Sci. Papers Coll. Gen. Ed. Univ. Tokyo 3, 115–130 (1953).

Let U_1 and U_2 be two linear subspaces of dimensions r_1, r_2 of an orthogonal n-dimensional vector space V, and let P_i be the orthogonal projection operator $V\rightarrow U_i$ $(i=1, 2)$. Then the orthogonal invariants of $A=P_1P_2P_1$ are invariants of U_1 and U_2 under the orthogonal group $O(n)$. The author shows that the ordered eigenvalues $0\leq\lambda_1\leq\cdots\leq\lambda_n\leq 1$ of A are a complete set of invariants, and proves that the mapping $\theta: M_{n,r_1}\times M_{n,r_2}\rightarrow E_n$, defined by $\theta(U_1, U_2)=(\lambda_1,\cdots,\lambda_n)$ is a homeomorphism. $M_{n,r}$ is the Grassmann manifold of r-dimensional subspaces in V; $\theta(M_{n,r_1}, M_{n,r_2})$ is a topological cube of dimension r, where $r=\min(r_1, r_2, n-r_1, n-r_2)$. The same problem is dealt with for the Stiefel manifolds of orthogonal frames, but no complete solution is given. There are no references to the literature, such as to J. A. Schouten [Der Ricci Kalkül, Springer, Berlin, 1924, p. 45], who gave some essential parts of the argument, and also refers to some earlier works.

A. Nijenhuis (Princeton, N. J.).

Lemoine, Simone. Sur les variétés riemanniennes localement déformables d'un espace complet. C. R. Acad. Sci. Paris 238, 2052–2053 (1954).

Two theorems are proved: (1) In a sphere on four dimensions, every locally deformable and complete V_3 must contain the antipode of each of its points. (2) In a V_n which is simply connected and either locally euclidean $(n>3)$, or locally hyperbolic $(n=4)$, every V_{n-1} which is locally deformable and complete must contain a region at infinity.

C. B. Allendoerfer (Seattle, Wash.).

***Fenchel, W. On curvature and Levi-Cività's parallelism in Riemannian manifolds.** Convegno Internazionale di Geometria Differenziale, Italia, 1953, pp. 99–103. Edizioni Cremonese, Roma, 1954. 4000 Lire.

A geometrical interpretation is given for the integrands appearing in the generalized Gauss-Bonnet formula:

$$\omega_{n-1}=\int_{R^{n-1}} k_g dv+\int_{R^n} K_g dV.$$

The method involves a triangulation of R^{n-1} and an induced subdivision of R^n. The normals to R^{n-1} are displaced by Levi-Civita parallelism along well-chosen paths and a finite approximation to the above formula is obtained which yields the formula in the limit.

C. B. Allendoerfer.

***Willmore, Thomas J. Some properties of harmonic Riemannian manifolds.** Convegno Internazionale di Geometria Differenziale, Italia, 1953, pp. 141–147. Edizioni Cremonese, Roma, 1954. 4000 Lire.

A number of equivalent definitions of harmonic Riemannian spaces are stated and discussed. A definition is also given for a Riemannian manifold in the large, and it is shown that the Riemannian metrics naturally associated with connected compact two-point homogeneous spaces are harmonic. It is also shown that if a compact orientable manifold, with a non-flat harmonic metric, admits a one-parameter group of motions, then its first Betti number is zero. Finally it is conjectured that every harmonic metric defined over a compact orientable manifold is symmetric in the sense of Cartan.

A. G. Walker (Liverpool).

Ōtsuki, Tominosuke. Structure of a Riemann space. Proc. Japan Acad. **29**, 475–477 (1953).

The author studies Riemann spaces whose Ricci tensor satisfies the conditions $R_i^{\ k}R_k^{\ j}=(n-1)^{-1}RR_i^{\ j}$, $R^i_{\ i,k}=0$, where the comma denotes covariant differentiation. It is proved that such a space is either an Einstein space with zero scalar curvature or the product space of an Einstein space and a straight line. *S. Chern* (Chicago, Ill.).

Kurita, Minoru. On the isometry of a homogeneous Riemann space. Tensor (N.S.) **3**, 91–100 (1954).

The significant theorem of the paper gives the types of homogeneous Riemann spaces of dimension n whose isotropy group induces in the tangent space a group of linear transformations which keeps every single vector of an $(n-k)$-dimensional subspace fixed, and gives the full group of rotations in its orthogonal complement. There appear to be four types, one of which involves a locally flat manifold, while two others involve spaces of constant curvature.
A. Nijenhuis (Princeton, N. J.).

***Lichnerowicz, A. Sur les groupes d'holonomie des variétés riemanniennes et kähleriennes.** Convegno Internazionale di Geometria Differenziale, Italia, 1953, pp. 33–44. Edizioni Cremonese, Roma, 1954. 4000 Lire.

This is an exposition of recent work by A. Borel and the author [C. R. Acad. Sci. Paris **234**, 1835–1837; **235**, 12–14 (1952); these Rev. **13**, 986; **14**, 89] on the groups of holonomy of Riemannian and pseudo-Kählerian manifolds (p-K manifolds), covering the definition and basic properties of groups of holonomy of Riemannian manifolds, with applications to p-K manifolds, including the characterisation of these by means of the groups, properties of reducible p-K manifolds, and p-K manifolds with Ricci-curvature zero.
W. V. D. Hodge (Cambridge, England).

Vrănceanu, G. Sur les groupes de mouvement d'un espace de Riemann à quatre dimensions. Acad. Repub. Pop. Romîne. Stud. Cerc. Mat. **4**, 121–153 (1953). (Romanian. Russian and French summaries)

In the first section of this paper the Riemannian V_4 with a G_4 as stability group are determined, reestablishing a result of S. Medici [Ann. Scuola Norm. Super. Pisa **10**, no. 3 (1908)]. In the second section it is shown that the only V_4 with a group of motions G_8 are the symmetric spaces of Cartan already investigated by I. P. Egorov [Doklady Akad. Nauk SSSR (N.S.) **66**, 793–796 (1949); these Rev. **11**, 211]. It is also shown that if we pass from a space V_n $(n>1)$ of constant curvature with group $G_{n(n+1)/2}$ to a space V_n of variable curvature, the number of parameters decreases with $n-1$, except in the case of the symmetrical V_4, where the decrease is $n-2=2$ and of the V_2, where the decrease is $n=2$ [this improves on a result by G. Fubini, Ann. Mat. Pura Appl. (3) **9**, 33–90 (1903)]. In the third section we find a discussion of the V_4 with a G_3, a G_2 and a G_1. One of the results is that the only V_4 with stability group

$$x^1\frac{\partial f}{\partial x^2}-x^2\frac{\partial f}{\partial x^1},\quad x^1\frac{\partial f}{\partial x^3}-x^3\frac{\partial f}{\partial x^1},\quad x^2\frac{\partial f}{\partial x^3}-x^3\frac{\partial f}{\partial x^2},$$

which have a transitive G_7 as their complete group of motions, have a line element which can be written in the form $ds^2=d\sigma^2+(dx^4)^2$, where $d\sigma^2$ represents a metric of constant curvature in x^1, x^2, x^3. *D. J. Struik.*

***Yano, Kentaro. Groups of motions and groups of affine collineations.** Convegno Internazionale di Geometria Differenziale, Italia, 1953, pp. 229–233. Edizioni Cremonese, Roma, 1954. 4000 Lire.

The author, first, gives a brief account of the known results concerning Riemannian spaces admitting a "high"-dimensional group of motions. This shows the rôle which the isotropic subgroup plays in this kind of problem. The author then generalizes this idea to study the group G of affine collineations of an n-dimensional symmetric affinely connected space M when the dimension of the isotropic subgroups is greater than n^2-2n+6. All the possible isotropic subgroups are enumerated, and the curvature tensors of M are determined. This has some contact with works of Egorov [Doklady Akad. Nauk SSSR (N.S.) **73**, 265–267 (1950); these Rev. **12**, 636]. *H. C. Wang.*

Cossu, Aldo. Su alcune connessioni affini localmente associate ad una assegnata connessione asimmetrica. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **16**, 193–198 (1954).

Given a non-symmetric linear connection $L^k_{\ ij}$, the question arises whether it is possible to construct a linear connection $\bar L^k_{\ ij}$ such that the 3-dimensional surfaces E_3 of geodesics of L and $\bar L$ issuing from a point p, with the same tangent 3-planes have second-order contact, and such that some other conditions are satisfied in addition. Some of the conditions for which the answer is in the affirmative are: (1) $\bar L(p)=L(p)$; (2) vectors ξ that stay tangent to an E_3 for displacements around infinitesimal circuits of E_3 by means of L do the same for $\bar L$; (3) L and $\bar L$ displace volumes in the same manner; (4) $\bar L$ is locally integrable, i.e. its curvature tensor vanishes at p; (5) L and $\bar L$ have the same symmetric parts; (6) the parallel transport of directions is locally integrable. *A. Nijenhuis* (Princeton, N. J.).

Cossu, Aldo. Alcune osservazioni sul confronto di due connessioni affini. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **13**, 189–198 (1954).

Let $L^i_{\ hk}(x)$ and $\bar L^i_{\ hk}(x)$ be the parameters which determine two affine connections in an X_n. Given a vector ξ'^i at the point x^i+dx^i, let ξ^i and $\bar\xi^i$ be the two vectors at the point x^i which are parallel to ξ'^i with respect to the first and the second connections respectively. There is then between ξ^i and $\bar\xi^i$ the relation

$$(1)\qquad \bar\xi^i=(\delta_h^{\ i}+\psi_{hk}^{\ \cdot\cdot i}dx^k)\xi^h \quad\text{with}\quad \psi_{hk}^{\ \cdot\cdot i}=\bar L^i_{hk}-L^i_{hk}.$$

The author studies configurations which are invariant under the homography (1). He also determines directions and vectors in relation to which the two given connections determine the same law of parallel transport.
E. T. Davies (Southampton).

Nijenhuis, Albert. A theorem on sequences of local affine collineations and isometries. Nieuw Arch. Wiskunde (3) **2**, 118–125 (1954).

The author proves the following theorem: "Let M be an analytic manifold with an analytic affine connection. Let p_i, $i=1, 2, \cdots$, be a convergent sequence of distinct points of M, $\lim p_i=p$, and for every p_i let there be a mapping φ_i which carries a neighbourhood of p_i into a neighbourhood of p in such a way that the connection is invariant under φ_i. Then M admits a local one-parameter group of affine motions in a neighbourhood of p with non-vanishing translation part at p. If M is a Riemannian space, and if the φ_i are isom-

etries, then there is a local one-parameter group of (metric) motions in a neighbourhood of p."

The main features of the proof run as follows. Denote by $\Phi(p_i)$ any one of the tensors $R_{\omega\mu\lambda}{}^{\nu}$, $S_{\lambda\mu}{}^{\nu}$ (curvature and torsion tensors) or any one of their covariant derivatives at p_i. Let p_ip be a geodetic arc through p_i and p of length s_i and tangential vector v^{λ} at p. Transform $\phi(p_i)$ by parallelism along p_ip into $\Phi_i(p)$. Then there is a linear non-singular transformation A_i which carries $\Phi_i(p)$ into $\Phi(p)$:

$$(1)\qquad v^{\mu}\nabla_{\mu}\Phi=\lim\frac{A_i^{-1}\Phi-\Phi}{s_i},\quad \lim s_i=0\quad (\phi\overset{\mathrm{def}}{=}\phi(p)).$$

Let B be any linear transformation and B^* the operator defined as follows:

$$(2)\qquad B^*\phi=\lim\frac{(I+\epsilon B)\cdot\Phi-\Phi}{\epsilon}.$$

The author proves that according to (1) there is a transformation V such that

$$(3)\qquad v^{\mu}\nabla_{\mu}\Phi=V^*\Phi.$$

The equations (3) show that v^{ν} and $V_{\lambda}{}^{\nu}$ satisfy the integrability conditions of the equations of motion

$$(4)\qquad \nabla_{\mu}v^{\nu}=V_{\mu}{}^{\nu}-2S_{\mu\lambda}{}^{\nu}v^{\lambda},\quad \nabla_{\mu}V_{\lambda}{}^{\nu}=-v^{\nu}R_{\omega\mu\lambda}{}^{\nu}.$$

In the case of the isometric space, Φ is the metric tensor and the equations (3) yield the Killing equation.

V. Hlavatý (Bloomington, Ind.).

***Kuiper, N. H. On convex locally-projective spaces.** Convegno Internazionale di Geometria Differenziale, Italia, 1953, pp. 200–213. Edizioni Cremonese, Roma, 1954. 4000 Lire.

A locally-projective space is a connected manifold X together with a system of mutually compatible maps into the real projective space P^n of the same dimension, or more conveniently into the universal covering space S^n of P^n. Further, X is called convex if every homotopy class of curves connecting any two of its points contains a locally straight line. The author proves the following three theorems for convex, locally-projective spaces X. 1. If X is of dimension >1, then the universal covering space is S^n or a convex domain X^* in S^n. If, further, X is not simply connected and $X^*\neq S^n$, then X^* is contained in a hemisphere of S^n and is projectively equivalent to a convex set in affine space. 2. If X is not simply connected and has a finite fundamental group, it is compact. 3. If X is a compact surface, it is projectively equivalent to the sphere, elliptic plane, locally Euclidean Klein bottle or locally affine torus (with universal covering space the affine plane, half-plane, or quarter plane) if it is homeomorphic to one of these; otherwise it is projectively equivalent to a locally hyperbolic surface, in which case it admits a Hilbert metric compatible with the projective structure and has as its universal covering space a strictly convex bounded domain whose boundary contains no twice differentiable interval. *W. M. Boothby.*

Mikami, Misao. A projective theory of the manifold of surface-elements. Mem. Fac. Engrg. Kyushu Univ. 13, 259–303 (1953).

The K-planes at each point x of an n-dimensional manifold M form a fibre bundle B (with projection mapping π) in which each fibre $\pi^{-1}(x)$ over a point $x\in M$ consists of the K-dimensional linear subspaces p of the tangent space to M at x. Then B is taken as the base space for another fibre bundle with n-dimensional projective spaces P_n as its fibres. In this latter fibre bundle a projective linear connection is given with the following properties. (1) In each $P_n(p)$ one point is marked as a "point of contact", which is identified with $\pi(p)$. (2) Every tangent line element dp of B is represented in $P_n(p)$ by an infinitesimal displacement of the contact point in such a way that d_1p and d_2p give the same displacement if and only if $\pi(d_1p)=\pi(d_2p)$. (3) Displacements by parallelism along paths within each fibre $\pi^{-1}(x)$ map contact points into each other. The projective connection is studied in great detail; semi-natural frames, natural frames, torsion, curvature are introduced, and their invariance under the significant transformations is investigated. Then follows the determination of a projective connection from a class of projectively related affine connections, by first restricting the class of admissible projective connections through the introduction of some invariant conditions. Finally, the projective connection is found to be related to an affine connection in a space of dimension one higher. The methods of the paper are similar to those of K. Yano [Thèse, Paris, 1938 = Ann. Sci. Univ. Jassy. Sect. I. **24**, 395–464 (1938); Proc. Phys.-Math. Soc. Japan (3) **24**, 9–25 (1942); these Rev. **7**, 330]; and ample reference is made to papers by Veblen [J. London Math. Soc. **4**, 140–160 (1929)] and T. Y. Thomas [Math. Z. **25**, 723–733 (1926)]. The author does not use the terminology of fibre bundles, which was used here for briefness of formulation.

A. Nijenhuis (Princeton, N. J.).

***Kawaguchi, Akitsugu. On the theory of non-linear connections.** Convegno Internazionale di Geometria Differenziale, Italia, 1953, pp. 27–32. Edizioni Cremonese, Roma, 1954. 4000 Lire.

The author develops the idea of a non-linear connection which appeared for special cases in Friesecke [Math. Ann. **94**, 101–118 (1925)] and Bortolotti [Ann. of Math. (2) **32**, 361–377 (1931)]. With each point of an n-dimensional manifold X_n there is associated an N-dimensional vector space E_N. A vector v^I of the E_N at one point x of X_n is related to a vector of the E_N attached to a neighboring point $x+dn$ by means of a law of connection to which corresponds an absolute differential defined by

$$\delta v^I=dv^I+\omega_i{}^I(x,v)dx^i$$

in which the functions $\omega_i{}^I$ involve not only the point coordinates x, but also the vector v which is "transported by parallelism". The functions $\omega_i{}^I$ are homogeneous of the first degree in v^I. The absolute differential of a contravariant tensor of any order is defined in a similar manner. To introduce a corresponding notion for a covariant vector the author makes use of the well-known relation between a covariant vector and a contravariant $(N-1)$-vector. The theory of Finsler spaces is shown to fit naturally into the theory developed. *E. T. Davies* (Southampton).

Martinelli, Enzo. Qualche proprietà geometrica nelle varietà a struttura complessa. Atti Accad. Ligure 9 (1952), 89–98 (1953).

Soit V une variété analytique complexe. Le plan affine réel sous-tendu par un vecteur tangent complexe est muni d'une orientation naturelle et est dit facette plane caractéristique (en abrégé f. p. c.). En un point x de V, les fonctions coordonnées $z^1,\cdots,z^n$ définissent une application d'un voisinage I_x de x sur un voisinage U de l'origine de l'espace numérique complexe. L'image, dans I_x, de la métrique euclidienne de U est une métrique riemanniene; toute mé-

trique sur V, qui provient, sur chaque I_s, d'une telle métrique, par transformation pseudo-conforme, est dite pseudo-conforme. Les métriques pseudo-conformes ne sont autres que les métriques hermitiennes. De plus, pour qu'une métrique hermitienne soit kählérienne, il faut et il suffit que, par transport parallèle, l'angle de deux vecteurs tangents d'une f. p. c. soit conservé. On montre que l'angle de deux vecteurs d'une même f. p. c. ne dépend pas de la métrique pseudo-conforme, et l'opérateur C de la théorie des formes différentielles sur les variétés analytiques complexes [voir, par ex., A. Weil, Comment. Math. Helv. **20**, 110–116 (1947); ces Rev. **9**, 65] détermine une rotation de $\pi/2$, du faisceau des vecteurs tangents d'une f. p. c. Considérons les couples de directions directement perpendiculaires s, n appartenant la même f. p. c.; alors, si u et v sont des fonctions réelles de point sur V liées par l'équation $du/ds=dv/dn$, la fonction $f=u+iv$ est holomorphe sur V. Une généralisation de cette équation à des fonctionnelles de cycles à p dimensions, conduit à la notion d'intégrale quasi-analytique [voir l'article de l'auteur dans Atti 4° Congresso Un. Mat. Ital., Taormina, 1951, v. 2, Edizioni Cremonese, Roma, 1953, pp. 398–406; ces Rev. **15**, 154]. *P. Dolbeault* (Paris).

Garabedian, P. R., and Spencer, D. C. A complex tensor calculus for Kähler manifolds. Acta Math. **89**, 279–331 (1953).

Sur une variété analytique complexe, l'opérateur différentiel d se décompose en la somme de deux opérateurs ∂ et $\bar\partial$ de types respectifs $(1, 0)$ et $(0, 1)$; si la variété est munie d'une métrique kählérienne, l'opérateur codifférentiel δ se décompose en $\mathfrak{b}$ et $\bar{\mathfrak{b}}$ de types respectifs $(0, -1)$ et $(-1, 0)$. Cet article a été écrit avant la parution de l'article de W. V. D. Hodge [Proc. Cambridge Philos. Soc. **47**, 504–517 (1951); ces Rev. **13**, 75] dans lequel ces opérateurs ont été définis pour la première fois; il a pour but d'étendre aux opérateurs ∂, $\bar\partial$, $\mathfrak{b}$, $\bar{\mathfrak{b}}$ sur les variétés kählériennes, des résultats connus pour les opérateurs d et δ sur les variétés riemanniennes. Les auteurs donnent, d'abord, des résultats valables pour les variétés (resp. variétés à bord compactes) riemanniennes quelconques: si B est une variété à bord qui possède une singularité fondamentale pour l'opérateur laplacien Δ, il existe une forme de Green sur B qui permet de résoudre un problème aux limites pour les formes différentielles C^∞; condition nécessaire et suffisante pour que B possède une singularité fondamentale; preuve de l'existence d'une singularité fondamentale pour les champs harmoniques (formes fermées et cofermées) sur une variété riemannienne C^∞ par une méthode différente de celle employée par K. Kodaira [Ann. of Math. (2) **50**, 587–665 (1949); ces Rev. **11**, 108] et qui semble plus proche de la méthode classique utilisée sur les surfaces de Riemann. Soit maintenant M une variété kählérienne de dimension complexe k. L'espace P (resp. L) des formes de type $(p, 0)$ (resp. des p-formes) de carré scalaire fini, est la somme de l'espace des formes analytiques (resp. harmoniques) et de l'adhérence, dans P (resp. L), de l'espace des formes $\mathfrak{b}\psi^{p,1}$ (resp. $\Delta\chi$). Un courant T est dit analytique si, ou bien il est de type $(p, 0)$ et $\bar\partial T=0$, ou bien il est de type $(k, k-p)$ et $\bar{\mathfrak{b}}T=0$; il est alors harmonique; on détermine une décomposition de l'espace des courants qui convergent sur toute forme C^∞, de carré scalaire fini; cette décomposition comprend l'espace des courants analytiques ou celui des formes harmoniques. Sur une variété kählérienne à bord, on étudie la relation entre les p-formes harmoniques et les p-formes analytiques, surtout dans le cas $p=0$, et on donne une formule de Cauchy pour les formes complexes satisfaisant à $\bar\partial\varphi=0=\mathfrak{b}\varphi$(ou $\partial\varphi=0=\bar{\mathfrak{b}}\varphi$). Puis on résoud le problème aux limites suivant pour une sous-variété à bord B d'un espace kählérien compact donné: trouver une forme φ de type (p, k) sur B telle que: $\bar{\mathfrak{b}}\partial\varphi=0$, $\varphi=\bar{\mathfrak{b}}\psi$ dans B et que la restriction de φ à la frontière de B soit donnée; on montre qu'il existe une solution unique; la méthode de démonstration est parallèle à celle de Garabedian et Spencer [Trans. Amer. Math. Soc. **73**, 223–242 (1952); ces Rev. **14**, 462], une fois qu'une singularité fondamentale pour $\bar{\mathfrak{b}}\partial$ a été définie; on résoud de la même façon le problème dual. La solution du second problème et d'un problème voisin du premier permet de définir les formes de Neumann N_p et de Green G_p dans B et le noyau reproduisant des formes ∂ et $\bar{\mathfrak{b}}$-fermées de type $(p, 0)$; ces solutions s'expriment à l'aide de formes déduites de G_p et de N_p. On construit ensuite la singularité fondamentale pour les formes ∂ et $\bar{\mathfrak{b}}$-fermées sur B; enfin, on construit, sur une variété kählérienne compacte, un courant de type donné, harmonique ou ∂ et $\bar{\mathfrak{b}}$-fermé, dont la partie singulière est donnée au voisinage de chaque point.
P. Dolbeault (Paris).

Kodaira, K. On cohomology groups of compact analytic varieties with coefficients in some analytic faisceaux. Proc. Nat. Acad. Sci. U. S. A. **39**, 865–868 (1953).

Cette note et les deux suivantes seront désignées respectivement par I, II et III; on y emploiera les mêmes notations. Soit V une variété analytique complexe et F un espace fibré analytique de base V, à fibre vectorielle de dimension un (i.e. la fibre est le corps des nombres complexes C), de groupe structural le groupe multiplicatif des complexes agissant sur C. Si F est défini, sur un recouvrement $\{U_j\}$ de V, par le système de changements de cartes locales $\{f_{jk}\}$, une forme différentielle φ à coefficients dans F est donnée par un système de formes φ_j sur U_j telles que $\varphi_j=f_{jk}\varphi_k$ sur $U_j\cap U_k$. Soit $\Omega^p(F)$ le faisceau des germes de p-formes holomorphes sur V, à coefficients dans F. Désignons par $\bar\partial$ la partie de type $(0, 1)$ de l'opérateur de différentiation d. Alors, le $q$ième groupe de cohomologie $H^q(V, \Omega^p(F))$ de la variété analytique complexe V, à coefficients dans le faisceau analytique $\Omega^p(F)$, est isomorphe à l'espace vectoriel de $\bar\partial$-cohomologie des formes différentielles C^∞ de type (p, q) sur V, à coefficients dans F [généralisation, due à J.-P. Serre, du cas où $\Omega^p(F)$ est le faisceau des germes de p-formes holomorphes ordinaires sur V; cf. P. Dolbeault, C. R. Acad. Sci. Paris **236**, 175–177 (1953); ces Rev. **14**, 673]. Supposons maintenant V compacte; munissons-la d'une métrique hermitienne arbitraire mais fixe et soit δ la partie de type $(0, -1)$ de l'opérateur de codifférentiation. On dit qu'une forme différentielle φ, C^∞, à coefficients dans F, est harmonique si $\bar\partial\varphi=0=\delta\varphi$, ce qui équivaut à $\Box\varphi=0$ (où $\Box=\bar\partial\delta+\delta\bar\partial$) [dans le cas où F est trivial, l'opérateur $\Box$ a été étudié par Garabedian et Spencer dans le mémoire analysé ci-dessus]. On a une formule de décomposition comme dans le cas des formes harmoniques classiques, d'où un isomorphisme de l'espace de $\bar\partial$-cohomologie des formes de type (p, q) sur V et de l'espace des formes harmoniques de type (p, q); de plus, l'opérateur $\Box$ étant elliptique, ce dernier espace est de dimension finie. En particulier, $H^q(V, \Omega^p(F))$ est de dimension finie. *P. Dolbeault.*

Kodaira, K., and Spencer, D. C. Groups of complex line bundles over compact Kähler varieties. Proc. Nat. Acad. Sci. U. S. A. **39**, 868–872 (1953).

Considérons les espaces fibrés analytiques complexes F définis dans I. Le groupe multiplicatif des complexes C^* étant abélien, il est clair que les classes d'espaces fibrés F isomorphes constituent un groupe abélien $\mathfrak{F}$. Le groupe $\mathfrak{F}$ est d'ailleurs isomorphe au groupe des classes d'espaces fibrés principaux associés aux espaces F. Le groupe des classes d'équivalence linéaire des diviseurs de V est isomorphe à un sous-groupe de $\mathfrak{F}$. Soit $\mathfrak{C}^*$ le faisceau des germes d'applications analytiques complexes de V dans C^*; Ω^0 le faisceau des germes de fonctions holomorphes, Z le faisceau constant des entiers sur V. La suite $0 \to Z \to \Omega^0 \to \mathfrak{C}^* \to 0$, où la seconde flèche désigne l'injection et la troisième l'application $\varphi \to \exp 2\pi i \varphi$, est exacte; elle définit la suite exacte de cohomologie

$$H^1(V; Z) \xrightarrow{f} H^1(V; \Omega^0) \xrightarrow{g} H^1(V, \mathfrak{C}^*) \xrightarrow{h} H^2(V, Z) \xrightarrow{j} H^2(V, \Omega^0).$$

Il y a une correspondance biunivoque entre les éléments de $H^1(V, \mathfrak{C}^*)$ et les classes d'espaces fibrés F. L'image c de F dans $H^2(V; Z)$ est la classe caractéristique de F. Désignons par $H^{p,q}(V)$ l'espace vectoriel des formes harmoniques de type (p, q). D'après le théorème de Hodge, on peut définir la partie harmonique Hc de c; or $H^2(V, \Omega^0)$ est isomorphe à $H^{0,2}(V)$; l'image de h étant le noyau de j, c'est le sous-groupe $H^2_{(1,1)}(V, Z)$ de $H^2(V; Z)$ formé des éléments c tels que Hc soit de type $(1, 1)$. Soit $\mathfrak{P}$ le noyau de h; alors $\mathfrak{F}/\mathfrak{P} \approx H^2_{(1,1)}(V; Z)$ et $\mathfrak{P}$ est isomorphe au conoyau de f; on sait qu'il existe un isomorphisme μ: $H^1(V, \Omega^0) \to H^{0,1}(V)$; soit $\mathfrak{I}$ l'image de $\mu \circ f$, alors $\mathfrak{P} \approx H^{0,1}(V)/\mathfrak{I}$ qui est un tore complexe dont la dimension est la moitié du premier nombre de Betti de V; le groupe $\mathfrak{P}$ peut être muni de la structure analytique complexe du tore $H^{0,1}(V)/\mathfrak{I}$. Généralisant la terminologie en usage lorsque V est algébrique, on appelle $\mathfrak{P}$ la variété de Picard de V. Théorème: La variété de Picard $\mathfrak{P}$ de V est un sous-groupe de $\mathfrak{F}$ et $\mathfrak{F}/\mathfrak{P} \approx H^2_{(1,1)}(V; Z)$. [La méthode de démonstration exposée, légèrement différente de celle des auteurs, est due à J.-P. Serre.]

P. Dolbeault (Paris).

Kodaira, K., and Spencer, D. C. Divisor class groups on algebraic varieties. Proc. Nat. Acad. Sci. U. S. A. **39**, 872–877 (1953).

Soit V une variété algébrique projective, non singulière. Désignons par $\{D\}$ l'espace fibré F défini par un diviseur D de V (voir II). Théorème 1: Pour tout espace fibré F sur V, à fibre vectorielle de dimension un, il existe un diviseur D de V, tel que $\{D\} = F$. Démonstration (par récurrence sur la dimension n de V): supposons d'abord $n \geq 2$. Soient S une section hyperplane générale fixée de V et S_m une hypersurface section générale d'ordre m de V. On désigne par $F \cdot S$ la restriction de F à S et par $\Omega(F \cdot S)$ le faisceau, sur S, des germes de sections analytiques de $F \cdot S$. Soit $\{U_j\}$ un recouvrement de V suffisamment fin; un diviseur D de V est défini, dans chaque U_j par une fonction méromorphe $R_j(z; D)$; soit $\zeta_j(z)$ une section locale de l'espace fibré F, au-dessus de U_j. On a la suite exacte

$$0 \to \Omega(F - \{S\}) \xrightarrow{i} \Omega(F) \xrightarrow{r} \Omega(F \cdot S) \to 0,$$

où i désigne l'injection définie par $\zeta_j(z) \to R_j(z; S)\zeta_j(z)$ ($z \in U_j$) et r la restriction. Posons $F_m = F + \{S_m\} = F_{m-1} + \{S\}$; on a donc la suite exacte

$$(1) \qquad 0 \to \Omega(F_{m-1}) \xrightarrow{i} \Omega(F_m) \xrightarrow{r} \Omega(F_m \cdot S) \to 0.$$

D'après l'hypothèse de récurrence, il existe un diviseur $\bar{D}$ sur S tel que $\{\bar{D}\} = F \cdot S$ sur S. Pour $m \geq m_0$, le système linéaire complet $|\bar{D} + S_m \cdot S|$ est suffisamment ample, alors, il résulte d'un théorème de Lefschetz qui relie la cohomologie de V à celle d'une section hyperplane de V que

$$H^1(S; \Omega(F_m \cdot S)) = H^1(S; \Omega(\bar{D} + S_m \cdot S)) = 0.$$

Les groupes de cohomologie $H^q(V, \Omega(F_m))$ sont de dimension finie d'après I; il résulte alors, de la suite exacte de cohomologie définie par (1) pour tous les $m \geq m_0$, que $\dim H^0(V, \Omega(F_m)) \to +\infty$ quand $m \to +\infty$. Donc, pour m suffisamment grand, l'espace fibré F_m a une section analytique globale définie par $\zeta_j(z)$ sur U_j, différente de la section 0. Soit D_m le diviseur de V défini par $\zeta_j(z)$ dans chaque U_j; on a $F = \{D_m - S_m\}$. A l'aide de la fin du raisonnement précédent, on montre l'existence d'un diviseur D tel que $\{D\} = F$, dans le cas $n = 1$; d'où le théorème 1. La classe caractéristique de l'espace fibré $\{D\}$ est associée, dans la dualité des variétés, à la classe d'homologie entière du cycle défini par le diviseur D. Soit $H^{(1,1)}_{2n-2}(V; Z)$ le sous-groupe des classes d'homologie de $H_{2n-2}(V; Z)$ dont les parties harmoniques sont de type $(1, 1)$ et soient $\mathfrak{G}$ le groupe additif des diviseurs de V, $\mathfrak{G}_a$ (resp. $\mathfrak{G}_r$) le sous-groupe de $\mathfrak{G}$ formé des diviseurs dont la classe d'homologie entière (resp. rationnelle) est nulle, $\mathfrak{G}_l$ le sous-groupe de $\mathfrak{G}_a$ des diviseurs linéairement équivalents à zéro. Du théorème 1 résultent: $\mathfrak{G}/\mathfrak{G}_l = \mathfrak{F}$, puis: théorème 2: $\mathfrak{G}_a/\mathfrak{G}_l = \mathfrak{P}$; théorème 3: $\mathfrak{G}/\mathfrak{G}_a = H^{(1,1)}_{2n-2}(V; Z)$; théorème 4: pour qu'un $(2n-2)$-cycle entier d'une variété algébrique de dimension n, soit homologue à un diviseur, il faut et il suffit que sa partie harmonique soit de type $(1, 1)$; théorème 5: le groupe quotient $\mathfrak{G}_r/\mathfrak{G}_a$ est dual du premier groupe de torsion de V. Le théorème 4 est le critère de Lefschetz-Hodge [Hodge, The theory and applications of harmonic integrals, Cambridge, 1941, pp. 214–216; ces Rev. **2**, 296]; les théorèmes 2 et 5 sont les théorèmes de dualité d'Igusa [Amer. J. Math. **74**, 1–22 (1952); ces Rev. **13**, 680].

P. Dolbeault.

***Ehresmann, Charles. Introduction à la théorie des structures infinitésimales et des pseudo-groupes de Lie.** Géométrie différentielle. Colloques Internationaux du Centre National de la Recherche Scientifique, Strasbourg, 1953, pp. 97–110. Centre National de la Recherche Scientifique, Paris, 1953.

This lecture presents substantially the same material as one reviewed earlier, with the same title [Colloque de topologie et géométrie différentielle, Strasbourg, 1952, no. 11, Bibliothèque Nat. et Univ. Strasbourg, 1953; these Rev. **15**, 828].

H. Samelson (Ann Arbor, Mich.).

Aragnol, André. Classes caractéristiques et formes différentielles. C. R. Acad. Sci. Paris **238**, 2387–2389 (1954).

The author gives a new derivation of the results of the reviewer in which differential forms are defined which determine the Stiefel-Whitney characteristic classes. The method uses a series of fibre spaces defined over a complex and results in a considerable simplification of the derivation first given.

C. B. Allendoerfer (Seattle, Wash.).

Flanders, Harley. An extension theorem for solutions of $d\omega = \Omega$. Proc. Amer. Math. Soc. **5**, 509–510 (1954).

Let U and V be open sets in E_n such that $\bar{V} \subset U$ and U is connected and homologically trivial. Suppose that Ω is a differential form of class C^∞ on E_n such that $d\Omega = 0$, and that α is a form defined on U such that $d\alpha = \Omega$ on U. It is proved that there exists a form β on E_n such that $\beta = \alpha$ on V and $d\beta = \Omega$ on E_n.

C. B. Allendoerfer (Seattle, Wash.).

Yûjôbô, Zuiman. On a sufficient condition for a tensor to be harmonic. Proc. Japan Acad. **29**, 96–98 (1953).

Let α be a p-form on an n-dimensional Riemannian space M, with bounded coefficients, and suppose that α satisfies the conditions: (i) the coefficients of α are well defined and continuous on M, except on a set E_1 whose $(n-1)$-dimensional measure is zero; (ii) α is totally differentiable and satisfies $d\alpha=\delta\alpha=0$ on M, except at most on a sum E_2 of an enumerable infinity of sets whose $(n-1)$-dimensional measure is finite. The paper proves that in these circumstances α is harmonic on M. *W. V. D. Hodge.*

Tzou, K. H. Sur les champs vectoriel et pseudovectoriel. J. Phys. Radium (8) **15**, 559–562 (1954).

The author considers vector fields and pseudo-vector fields which satisfy second-order wave equations with a mass term present but do not satisfy any auxiliary condition such as the vanishing of the four-dimensional divergence. Such fields are decomposed into two parts, the first of which has a vanishing divergence and the second of which has a vanishing four-dimensional curl, that is, is the gradient of a scalar. It is shown that the first part which represents a particle of spin one is not coupled to the second part which represents a particle of spin zero by an interaction with a spinor field if the interaction energy is invariant and depends linearly on the vector field and its first derivatives. The interaction representation is discussed and is shown to be simple to determine in the absence of a supplementary condition. *A. H. Taub* (Urbana, Ill.).

***Rund, Hanno. On the geometry of generalised metric spaces.** Convegno Internazionale di Geometria Differenziale, Italia, 1953, pp. 114–121. Edizioni Cremonese, Roma, 1954. 4000 Lire.

The lemma of Ricci upon the vanishing of the covariant derivative of the metric tensor plays a fundamental role in classical Riemannian geometry as well as in Finsler geometry as developed by E. Cartan. The author's standpoint is to regard Finsler spaces as locally Minkowskian rather than locally Euclidean so that the lemma of Ricci is replaced by a more general result. In this lecture the author examines the extent to which a similar lemma can hold for Finsler spaces regarded as locally Minkowskian. He distinguishes between a relative covariant derivation which is suitable for dealing with special problems such as geodesic deviation and the theory of subspaces, and an absolute covariant derivative which is more useful in dealing with properties of the space as such, in particular with curvature theory. He gives the most general form of the Ricci lemma in terms of absolute covariant derivation. *E. T. Davies.*

Gołąb, S. Über den Begriff der kovarianten Ableitung. Nieuw Arch. Wiskunde (3) **2**, 90–96 (1954).

The absolute derivative of a geometric object of the first class along a curve is required to satisfy the conditions: (I) It depends on the given object and its first derivative, on a Γ-object, and on the tangent to the curve; (II) the derivative is an object of the first class; (III) (optional) the derivative of Ω is an object of the same type as Ω. For the case of an object with one component in a one-dimensional space, the most general formulas for derivatives satisfying (I), (II) and also (I), (II), (III) are stated without proofs. Formulas are also given in this case for the absolute derivative with respect to a second parameter ξ such that $d\xi/dt$ is the tangent to the curve. *C. B. Allendoerfer.*

Gołąb, S. Sur la dérivée covariante des objets géométriques de deuxième classe. Ann. Polon. Math. **1**, 107–113 (1954).

Let ω be a geometric object of class 2 in an X_1. This paper deals with the problem of defining a covariant derivative $D\omega=F(\omega,\omega',\mu)$, μ being a geometric object of class 3 not depending on ω, such that $D\omega$ is a geometric object of class 2. It is shown that such a covariant derivative does not exist for the general pseudo-group $\xi^*=\varphi(\xi)$. It exists however at P if one restricts the coordinate transformations to those for which at P either $\alpha=\varphi'=1$ or $\alpha>0$. *J. Haantjes* (Leiden).

Yano, Kentaro, and Tashiro, Yoshihiro. Some theorems on geometric objects and their applications. Nieuw Arch. Wiskunde (3) **2**, 134–142 (1954).

A geometric object is defined as a set of components $\Omega^A(x)$ in each coordinate system such that when $x'^i=f^i(x)$,

$$\Omega'^A(x')=F^A\left(\Omega^B, x^i, x'^i, \frac{\partial x'^i}{\partial x^j}, \cdots, \frac{\partial^p x'^i}{\partial x^{j_1}\cdots\partial x^{j_p}}\right)=F^A(\Omega, x, x'),$$

where F^A has a transitive property. The Lie differential is

$$D\Omega^A=(x\Omega^A)\delta t=\Omega^A(x')-\Omega'^A(x').$$

Then $D\Omega'^A=(\partial F^A/\partial\Omega^B)D\Omega^B$; $D\Omega^A$ is a geometric object if and only if $F^A(\Omega, x, x')$ has the form

$$F^A=F_B{}^A(x, x')\Omega^B+F^A(x, x').$$

Then Ω^A is called linear; if $F^A(x, x')=0$, Ω^A is called linear homogeneous.

The paper is concerned with existence theorems for differential equations involving geometric objects. For example, if r linear homogeneous geometric objects $\Phi_a{}^A$ form a complete system (i.e. if they satisfy $x_b\Phi_a{}^A-x_a\Phi_b{}^A=c^e{}_{ba}\Phi_e{}^A$) with respect to an r-parameter group in n variables and the generic rank of the matrix $\|\xi_a{}^i\|$ of the group is $r\leqq n$, then the system of equations $x_a\Omega^A=\Phi_a{}^A$ is completely integrable. Similar theorems discuss the case when the rank of $\|\xi_a{}^i\|$ is less than r. The theorems are applied to derive generalizations of theorems due to Knebelman and Levine. *C. B. Allendoerfer* (Seattle, Wash.).

***Haantjes, J. On the notion of geometric object.** Convegno Internazionale di Geometria Differenziale, Italia, 1953, pp. 77–81. Edizioni Cremonese, Roma, 1954. 4000 Lire.

This paper presents a new definition of the concept of a geometric object. Let A be an ordered set of polynomials of the form:

$$f^\beta=a_\alpha{}^\beta z^\alpha+\frac{1}{2!}a^\beta{}_{\alpha_1\alpha_2}z^{\alpha_1}z^{\alpha_2}+\cdots+\frac{1}{s!}a^\beta{}_{\alpha_1\cdots\alpha_s}z^{\alpha_1}\cdots z^{\alpha_s},$$

where $|a_\alpha{}^\beta|\neq 0$. By substituting similar polynomials g^β of a set B into f^β and omitting terms of degree higher than s, a set of polynomials C is obtained which is called $AB=C$. This law of composition defines the group $L_n{}^s$. Let G be an effective topological transformation group on a space Y and $h(x)$ for $x\in X_n$ (a differentiable manifold) a homeomorphism of $L_n{}^s$ onto G which is continuous in x. Then the "geometric object bundle", B, is the fibre bundle with X_n as base space, Y as fibre, and G as group. Equivalence of two bundles B and B' is defined in a way somewhat different from Steenrod's definition of the equivalence of two fibre bundles. Transitive geometric objects are also defined. A classification of transitive geometric objects of class C^1 with fibre of dimension 1 is given. *C. B. Allendoerfer.*

NUMERICAL AND GRAPHICAL METHODS

Royal Society depository for unpublished mathematical tables. J. London Math. Soc. **29**, 504–512 (1954).
Royal Society depository for unpublished mathematical tables. Philos. Mag. (7) **45**, 599–609 (1954).

In 1951 the Royal Society established a depository for unpublished mathematical tables. It is hoped to form a valuable collection of unpublished tables that would otherwise be less generally available. The following is a list of the tables accepted up to the end of 1953 together with a brief description of each. The tables may be consulted in the Library of the Royal Society and photocopies may be prepared at the expense of those who desire them.

Extract from paper.

***Kühne, E. E. Tafel für r^{-3} mit dem Argument r^2 (r^2 von 1–100).** Deutsche Akademie der Wissenschaften zu Berlin. Veröffentlichungen des astronomischen Recheninstituts. Akademie-Verlag, Berlin, 1953. 35+11 pp. DM 10.50.

This table gives 5-decimal values of $r^{-3/2}$ for

$$r=1(.0001)1.3(.001)8.5(.01)50(.1)100.$$

Average first differences are given at the end of each line. There are 11 folding pages which contain interpolation tables. The table is lithoprinted from typescript.

D. H. Lehmer (Berkeley, Calif.).

Clunie, J. On Bose-Einstein functions. Proc. Phys. Soc. Sect. A. **67**, 632–636 (1954).

The function $G_k(\eta)$ is defined by

$$G_k(\eta)=\int_0^\infty \frac{x^k dx}{e^{x-\eta}-1}$$

if $\eta\leq 0$, and by the principal value of the integral when $\eta>0$. The author gives formulas (including asymptotic expansions for large $|\eta|$) to facilitate the numerical evaluation of $G_k(\eta)$, and gives 4D tables of $G_{1/2}(\eta)$ for $\eta=-3(.2)-0.6(.1)(.2)20$. [For the mathematical theory of this function see Truesdell, Ann. of Math. (2) **46**, 144–157 (1945); these Rev. **6**, 152.]

A. Erdélyi (Pasadena, Calif.).

Boesch, Walter. Die Berechnung einiger komplexer Werte auf einer Multipliziermaschine mit nur einem Multiplizierwerk. Z. Angew. Math. Physik **5**, 341–343 (1954).

Lotkin, Mark. Some problems solvable on computing machines. Comm. Pure Appl. Math. **7**, 149–158 (1954).

The paper gives a brief discription of the following problems, which have been solved on the automatic computing machines at the Ballistics Research Laboratories, Aberdeen, U. S. A., including sometimes an outline of their solution: (i) linear simultaneous equations and matrix inversion; (ii) determination of trajectories; (iii) symmetric air flows; and (iv) tabulation of the cumulative binomial probability function. A further sub-section indicates the problem of the growth of error in numerical integration of differential equations, and states a theorem to be published elsewhere.

D. C. Gilles (London).

Lemaître, G. Comment calculer? Acad. Roy. Belgique. Bull. Cl. Sci. (5) **40**, 683–691 (1954).

The author proposes that paper and pencil computing be replaced by paper and typewriter. Instead of using the available arabic numerals 0–9 it is proposed that the four letters i, j, k, l which lie within easy reach of the right hand be used to give a peculiar binary coded decimal representation in which

$$i=1,\quad j=2,\quad k=4,\quad l=8,$$

for example, and ijk means $i+j+k=7$. *D. H. Lehmer.*

Dahlquist, Germund. The Monte Carlo-method. Nordisk Mat. Tidskr. **2**, 27–43, 80 (1954). (Swedish. English summary)

Expository paper.

Cairns, S. S. Computational attacks on discrete problems. Proceedings of the symposium on special topics in applied mathematics, Northwestern University, 1953. Amer. Math. Monthly **61**, no. 7, part II, 29–31 (1954).

Teghem, J. Sur la régression polynomiale. De quelques polynômes orthogonaux. Bull. Inst. Agronom. et Stations Recherches Gembloux **21**, 160–170 (1953).

It is desired to fit a polynomial regression in which the independent variate X is equally spaced ($X=1, 2, \cdots, N$), except for $k-1$ gaps of a single X each. Formulas are presented to compute orthogonal polynomial values through the fourth degree when each gap is followed by h values of X, i.e. there are k groups of h values of X: $1, 2, \cdots, h$; $h+2, \cdots, 2h+1$; $2h+3, \cdots$. Finally, general methods are presented to construct orthogonal polynomial values of any degree when the gaps are arbitrarily distributed. Somewhat simpler methods are also given for symmetrically distributed gaps. An example for each of these last two cases is given for $N=9$ and 2 gaps (7 observations). *R. L. Anderson.*

Kikuta, Takashi. Convergence of iterative methods. Progress Theoret. Physics **10**, 653–672 (1953).

The convergency of iterative procedures in the algebraic case has been carefully studied by E. Schröder [Math. Ann. **2**, 317–365 (1870)], who also classified the procedures and developed methods for improving the rate of convergency. This investigation has been continued by D. R. Hartree [Proc. Cambridge Philos. Soc. **45**, 230–236 (1949); these Rev. **10**, 574] who considered also the more general case.

In the present paper, Kikuta has studied the convergence properties of some of the iterative methods used in eigenvalue problems and proposed alternative forms for improving the efficiency. In applications to scattering problems for the square-well potential, he has compared Schwinger's iterative method with Born's method of successive approximations and shown that, in the low energy region, the former is considerably superior, whereas, in the high-energy region, the latter is more favourable due to its simplicity.

P. O. Löwdin (Uppsala).

Varoli, Giuseppe. Alcune osservazioni sul metodo di iterazione per la risoluzione approssimata delle equazioni. Period. Mat. (4) **32**, 70–76 (1954).

Given the existence of a real root α of $x=\varphi(x)$, known to be unique and simple in the interval (a, b), the problem considered concerns the convergence to α of a real sequence $\alpha_1, \alpha_2, \cdots$, where $\alpha_{i+1}=\varphi(\alpha_i)$. Necessary and sufficient for the existence of an $\alpha_1\neq\alpha$ for which the sequence converges to α is the existence of an interval (a', b') containing α within which $|\varphi'(x)|<1$, but α_1 may or may not need to lie on this interval. The author considers graphically a number of cases that may arise. *A. S. Householder.*

Ludwig, Rudolf. Über Iterationsverfahren für Gleichungen und Gleichungssysteme. I. Z. Angew. Math. Mech. **34**, 210–225 (1954). (English, French and Russian summaries)

Following Hartree [Proc. Cambridge Philos. Soc. **45**, 230–236 (1949); these Rev. **10**, 574], the author defines an iteration $x_{i+1}=f(x_i)$ to be of order k at a root α of $x=f(x)$ in case $0=f'(\alpha)=\cdots=f^{(k-1)}(\alpha)\neq f^{(k)}(\alpha)$, and he notes that if $f(x)\equiv x-A(x)/B(x)$, then this is equivalent to $A^{(\nu)}(\alpha)=\nu B^{(\nu-1)}(\alpha)$ $(\nu=0, 1, \cdots, k-1)$, $A^{(k)}(\alpha)\neq kB^{(k-1)}(\alpha)$. From here he develops and tabulates a number of iterations of higher order, and concludes by showing that by forming a suitable linear combination of p iterations of given order one can increase the order by $p-1$. *A. S. Householder.*

Maehly, Hans J. Zur iterativen Auflösung algebraischer Gleichungen. Z. Angew. Math. Physik **5**, 260–263 (1954).

The slow convergence of Newton's formula for the successive approximation of the roots of an algebraic equation may be improved by a formula of Laguerre [Oeuvres, vol. I, Gauthier-Villars, Paris, 1898, pp. 87–103]. This formula with certain generalizations is useful in the computation of both real and complex roots. A method is also given for the computation of a further root after several roots have already been found, such that errors in the determination of the previous roots will not affect the accuracy with which the present one may be computed. *E. Frank.*

Lochs, Gustav. Die Konvergenzradien einiger zur Lösung transzendenter Gleichungen verwendeter Potenzreihen. Monatsh. Math. **58**, 118–122 (1954).

For the computation of the positive roots of the equations $\operatorname{tg} z=z$, $\operatorname{tg} z=2z$; $\operatorname{Cos} z \cos z=1$; $\operatorname{Cos} z\cos z=-1$, power series have been used provided such power series converge. Here the radius of convergence of these power series is determined. *E. Frank* (Chicago, Ill.).

Taylor, William C., Jr. A neglected method for resolution of polynomial equations. J. Franklin Inst. **257**, 459–464 (1954).

This paper describes the method of Daniel Bernouilli [Comment. Acad. Sci. Imp. Petropolitanae **3** (1728), 85–100 (1732)] for the solution of polynomial equations.

E. Frank (Chicago, Ill.).

Wagner, Harvey M. A partitioning method of inverting symmetric definite matrices on a card-programmed calculator. Math. Tables and Other Aids to Computation **8**, 139–143 (1954).

LaFara, Robert L. A method for calculating inverse trigonometric functions. Math. Tables and Other Aids to Computation **8**, 132–139 (1954).

Vietoris, L. Der Richtungsfehler einer durch das Adamssche Interpolationsverfahren gewonnenen Näherungslösung einer Gleichung $y'=f(x, y)$. Österreich. Akad. Wiss. Math.-Nat. Kl. S.-B. IIa. **162**, 157–167 (1953).

The method of Adams for the differential equation $y'=f(x, y)$ with $y(x_0)=y_0$ refers to pivotal points $x_i=x_0+ih$, $i=0, 1, \cdots$. It leads to an approximate value η_{n+1} for $y(x_{n+1})$ by extrapolation from the values $\eta_n, \eta_{n-1}, \cdots, \eta_{n-r}$. It is assumed, that $\eta_0, \eta_1, \cdots, \eta_r$ are given. The extrapolation can be described by means of a polynomial $\eta_{n+1}(x)$, such that $\eta_{n+1}(x_{n+1})=\eta_{n+1}$; $\eta_{n+1}(x)$ has no higher degree than $r+1$ and is defined by $\eta_{n+1}(x_n)=\eta_n$; $\eta'_{n+1}(x_i)=f(x_i, \eta_i)$ for $i=n, n-1, \cdots, n-r$. Estimates for the error of the method in the pivotal points have been given by various authors. The subject of this paper is to derive estimates for $q_{n+1}(x)=f(x, \eta_{n+1}(x))-\eta'_{n+1}(x)$ in the whole interval $x_n\leqq x\leqq x_{n+1}$. The author admits that the defining relations for the $\eta_{n+1}(x)$ are not exactly satisfied due to errors of computation. Instead he assumes that the relations

$$\eta_{n+1}(x_n)=\eta_n(x_n), \quad \eta'_{n+1}(x_i)=\eta_n'(x_i)$$

for $i=n, n-1, \cdots, n-r+1$ hold and that inequalities $|\eta_n'(x_n)-f(x_n, \eta_n)|<\rho$, $|\eta'_{n+1}(x_{n+1})-\eta_n'(x_{n+1})|<\sigma$ exist with ρ and σ covering the whole range of integration. (The reviewer does not quite understand these assumptions. He feels that the σ-inequality should read with x_n instead of x_{n+1} and that the subscript in the equalities for $\eta_n'(x_i)$ should run from $i=n-1$ to $i=n-r$.) Under suitable assumptions on the derivatives of $f(x, y)$, the results are some estimates for q_{n+1}, a typical one of them being

$$|q_{n+1}(x)|\leqq h^{r+1}C_1\cdot\max\left|\frac{d^{r+1}}{dx^{r+1}}q_{n+1}(x)\right|+C_2\rho+LhC_3\sigma$$

$$(x_{n-r}\leqq x\leqq x_n \text{ for max}),$$

with constants C_1, C_2, C_3, which depend on r only. L is a Lipschitz constant for $f(x, y)$. *H. Bückner.*

Vietoris, L. Der Richtungsfehler einer durch das Adamssche Interpolationsverfahren gewonnenen Näherungslösung eines Systems von Gleichungen

$$y_k'=f_k(x, y_1, y_2, \cdots, y_m).$$

Österreich. Akad. Wiss. Math.-Nat. Kl. S.-B. IIa. **162**, 293–299 (1953).

The author extends results of the paper reviewed above to systems of ordinary differential equations of first order. The new results are almost verbally the same, only the Lipschitz constant L of the previous paper is replaced by a matrix of such constants, the matrix operating on an error-vector $(\sigma_1, \sigma_2, \cdots, \sigma_m)$. Special attention is given to the case where the system stands for a single differential equation of higher order. *H. Bückner.*

Lozinskiĭ, S. M. On approximate solution of systems of ordinary differential equations. Doklady Akad. Nauk SSSR (N.S.) **97**, 29–32 (1954). (Russian)

Previous papers [same Doklady (N.S.) **92**, 225–228; **93**, 621–624 (1953); **94**, 17–19 (1954); these Rev. **15**, 473, 651, 793] gave conditions under which a solution of a system of ordinary differential equations would exist on a specified interval, and provided estimates of the deviation of an approximate solution from the true solution. The present theorems are an attempt to restate the results in a form that can be more readily applied. *A. S. Householder.*

Futterman, W., Osborne, E., and Saxon, David S. A numerical solution of Schrödinger's equation in the continuum. J. Research Nat. Bur. Standards **52**, 259–264 (1954).

Solutions of the Schrödinger equation for a neutron-proton system under the influence of a Yukawa potential for energies in the continuous spectrum have been determined by numerical methods. A numerical solution, obtained by outward integration from the origin by means of the Gauss-Jackson central-difference method, has been fitted to an asymptotic solution derived by a modification of the WKB procedure. The asymptotic phases as well as the wave functions are tabulated for the values 20, 50, 90,

120, and 150 MeV of the incident neutron energy. (In Table 3, l should stand instead of x.) *P. O. Löwdin.*

Berkofsky, Louis. The exact determination of the effective domain of dependence of a one-dimensional numerical prediction formula. Tellus 6, 165–169 (1954).

In any problem of numerical prediction of weather patterns by means of the physical equations governing air motion, the boundary conditions present difficulties unless observations are available over the whole globe. Fortunately, weather at a given place is influenced only to a minor extent by the conditions at a great distance. In the paper under review, this statement is made more quantitative. Starting with the one-dimensional barotropic forecast method first suggested by Charney and Eliassen, and assuming sinusoidal initial disturbance on the pressure field, it is shown that the error due to a limited amount of information depends on the region over which data are available according to an expression involving Lommel functions. In particular, the error due to omission of observations at a distance greater than 15° longitude from the point to which the forecast applies is less than 10%. *H. Panofsky.*

Mel'nik, S. I. Oscillating functions and their application to approximate solution of integral equations. Doklady Akad. Nauk SSSR (N.S.) **95**, 705–708 (1954). (Russian)

A function $f(p)$ which, together with its square, is summable over a region ω, is said to be an oscillating function in case ω can be subdivided into nonoverlapping regions ω_i such that $\int_{\omega_i} f(p)\, d\omega_p=0$ for every i. Now given an integral equation $u(p)-f(p)-\int_\omega k(p,q)u(q)\, d\omega_q=0$, one can undertake to approximate the solution u by a linear combination of n orthogonal functions θ_i: $u_n=\sum a_i\theta_i(p)$. When u_n replaces u the left member of the integral equation is a function $\psi_n(p, a_1, \cdots, a_n)$ and there are various conditions one can apply to ψ_n so as to determine the a_i. In this paper is considered in particular the requirement that the ψ_n be an oscillating function. A theorem and a corollary bound the error of an approximate solution, but the conclusions are obscured by evident misprints. *A. S. Householder.*

Biswas, S. N. Solutions of Heitler's integral equation by iteration method. Physical Rev. (2) **94**, 1767–1772 (1954).

C. Wagner has suggested a special iteration method for the integral equation $y(s)=f(s)-\lambda\int_0^1 K(s,t)y(t)dt$ by calculating iterations $y_0, y_1, \cdots$ from

$$y_{n+1}\left(1+\lambda\int_0^1 K(s,t)dt\right)=\left(f+y_n\lambda\int_0^1 K(s,t)dt-\lambda\int_0^1 K(s,t)y_n(t)dt\right).$$

This method is directly extended to Heitler's integral equation for radiation dampening in scattering processes,

$$R(k,k_0)=G(k,k_0)-\lambda\int_{\Omega'} G(k,k')R(k',k_0)d\Omega'$$

with Ω' as unit-sphere. For the scattering of negative mesons by protons and of positive mesons by protons one iteration step with an arbitrary initial approximation proved to be sufficient and in agreement with results of Goldberger, Hsüeh and Ma. Another application refers to the influence of radiation dampening on the scattering of positive mesons by protons using the pseudoscalar meson field with pseudovector coupling. The results show that radiation reaction prevents the indefinite increase of the scattering cross-section with increasing energy. The paper contains numerical data, diagrams, detailed formulas for the iteration, and it compares the results with experiments. *H. Bückner* (Schenectady, N. Y.).

Kitz, N., and Marchington, B. A method of Fourier synthesis using a standard Hollerith senior rolling total tabulator. Acta Cryst. 6, 325–326 (1953).

Hauptman, H., and Karle, J. Locating the principal maxima of a Fourier series. Acta Cryst. **6**, 469–473 (1953).

Lonsdale, K. The design and construction of the Manchester University digital computor. Electronic Engrg. **26**, 376–382 (1954).

Booth, A. D., and Holt, A. D. The selenium rectifier in digital computor circuits. Electronic Engrg. **26**, 348–355 (1954).

Jecklin, Heinrich. Lidstonesche Näherungsformel und Makehamsche Funktion. Bl. Deutsch. Ges. Versicherungsmath. **2**, no. 1, 61–70 (1954).

RELATIVITY

Scherrer, Willy. Grundlagen zu einer linearen Feldtheorie. Z. Physik **138**, 16–34 (1954).

In this paper the author formulates a classical field theory unifying gravitation and electromagnetism based on a set of four independent linear forms in a four-dimensional space. The first part of the paper is concerned with the tensor analysis associated with such a set of linear forms and the second part with the derivation of a set of field equations based on a variational principle involving a special Lagrangean. The gravitational field is determined from the linear forms by taking sums of products of the linear forms. The coefficients of the products in this sum are undetermined by the field equations and presumably are arbitrary. The electromagnetic field is determined from the derivatives of the linear forms and four constants associated with a set of four forms satisfying the field equations. *A. H. Taub* (Urbana, Ill.).

Taylor, N. W. An interpretation of the field tensor in the unified field theory. Australian J. Phys. **7**, 1–4 (1954).

The author assumes that the skew-symmetric part of the field tensor g_{ik} is a self-dual tensor. This assumption does not, however, allow for spherically symmetric electric and magnetic fields to exist in free space. *L. Infeld.*

Tonnelat, Marie-Antoinette. Application de la solution générale des équations $g_{\underset{+}{\mu}\underset{-}{\nu};\rho}=0$ à la détermination d'une connexion affine particulière. C. R. Acad. Sci. Paris **239**, 231–233 (1954).

The author has previously obtained [same C. R. **230**, 182–184 (1950); J. Phys. Radium (8) **12**, 81–88 (1951); **13**, 177–185 (1952); these Rev. **11**, 569; **13**, 79; **14**, 213] a general determination of the non-symmetrical affine connection in terms of the non-symmetrical tensor $g_{\mu\nu}$ by solving

the equations

$$g_{\underset{+}{\mu\nu};\rho}=0$$

proposed by Einstein. In this paper the components of the affine connection are calculated from the formulas previously obtained for the case when the $g_{\mu\nu}$ are spherically symmetric. *A. H. Taub* (Urbana, Ill.).

Papapetrou, A., und Urich, W. Zur Kohlerschen Formulierung der Gravitationstheorie. Ann. Physik (6) 14, 220–232 (1954).

At one time it was proposed by N. Rosen [Physical Rev. (2) 57, 147–150, 150–153, 154–155 (1940); these Rev. 1, 183] that, in discussing the Einstein field equations of general relativity, one should introduce, in addition to the metric tensor $g_{\mu\nu}$ of the curved space-time, a second metric tensor $\gamma_{\mu\nu}$ describing a flat space-time. This suggested the possible interpretation of the $g_{\mu\nu}$ as describing a gravitational field in a flat space. However, the $\gamma_{\mu\nu}$ did not appear in the field equations, but only in the auxiliary coordinate conditions. M. Kohler [Z. Physik 131, 571–602 (1952); 134, 286–305, 306–316 (1953); these Rev. 14, 416, 913] has considered more general field equations involving both tensors. The present work deals with a class of such equations. It is shown that, in general, they lead to the existence of static, spherically symmetric fields that cannot be generated by any possible matter distribution, an exception being the case of the Einstein field equations. The question of satisfying the principle of equivalence and the interpretation of the second metric tensor are discussed. *N. Rosen* (Haifa).

Chase, D. M. The equations of motion of charged test particles in general relativity. Physical Rev. (2) 95, 243–246 (1954).

The Lorentz equations of motion without radiation damping are derived for charged test particles from the field equations, $R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R=T_{\mu\nu}$, of ordinary general relativity. Though no restriction is placed on the strength of the background gravitational and electromagnetic fields, it is assumed that in the limit the mass and the charge of the particle go to zero, their ratio remaining constant. The method is a generalization of that used by Infeld and Schild for an ordinary (uncharged) particle. *L. Infeld.*

Werle, J. The problem of "equivalent" potentials in classical equations of motion. Bull. Acad. Polon. Sci. Cl. III. 1, 281–285 (1953).

The author shows that it is possible to approximate the classical relativistic equations of motion of a particle in certain cases by non-relativistic equations, provided in the latter a suitably chosen potential is used. Explicit expressions for this "equivalent" potential are obtained in several cases, such as that of an electrostatic field. *N. Rosen.*

MECHANICS

Bottema, O. The velocity distribution in a moving fixed body. Simon Stevin 30, 5–16 (1954). (Dutch)

A moving fixed body in R_3 determines a null system ($P\leftrightarrow$plane through P orthogonal to v_P). In the same way a moving fixed body in R_4 determines a null system (R_1 through the fixed point $0\leftrightarrow R_3$ through R_1 orthogonal to the velocity vectors of the points of R_1). The main properties of these null systems are deduced by geometrical means. *J. Haantjes* (Leiden).

Levitskiĭ, N. I. The kinematic synthesis of mechanisms according to Čebyšev. Akad. Nauk SSSR. Trudy Sem. Teorii Mašin i Mehanizmov 12, no. 48, 5–12 (1 plate) (1952). (Russian)

Geronimus, Ya. L. The dynamic synthesis of mechanisms according to Čebyšev. Akad. Nauk SSSR. Trudy Sem. Teorii Mašin i Mehanizmov 12, no. 48, 13–22 (1952). (Russian)

Rozovskiĭ, M. S. Selection of schemes of toothed reduction gears consisting of two differential three-member mechanisms. Doklady Akad. Nauk SSSR (N.S.) 96, 701–704 (1954). (Russian)

Two planetary differential gear systems, in each of which the two inputs and the output have a common axis of rotation, are combined to produce a more elaborate gear train in which the two inputs and the output are again coaxial. The additional parameters permit a selection, to suit a particular need, from a richer collection of possible rational transmission ratios. For any given set of gear pairs, the possible ratios are presented in a convenient graphical plot. *M. Goldberg* (Washington, D. C.).

Kreines, M., and Rozovskiĭ, M. The selection of schemes of toothed reduction gears consisting of three differential three-member mechanisms. Doklady Akad. Nauk SSSR (N.S.) 96, 1117–1120 (1954). (Russian)

The method in the paper reviewed above is extended to a combination of three planetary differential gear systems. *M. Goldberg* (Washington, D. C.).

Vonica, Ion. A remarkable case of an incomplete (inexact) solution to a classical problem of mechanics. Gaz. Mat. Fiz. Ser. A. 6, 164–170 (1954). (Romanian)

A problem of elementary statics: Two cylinders rest upon two smooth inclined planes and press against each other. *O. Bottema* (Delft).

***Signorini, Antonio. Sui moti rigidi sferici.** Convegno Internazionale di Geometria Differenziale, Italia, 1953, pp. 274–282. Edizioni Cremonese, Roma, 1954. 4000 Lire.

General considerations on the motion of a rigid body about a fixed point. Two problems are discussed: to determine the motion if the components of the angular velocity along fixed resp. moving axes are given as functions of the time. *O. Bottema* (Delft).

Block, H. D. A remark on integral invariants. Quart. Appl. Math. 12, 201–203 (1954).

In several text-books on classical mechanics the theorem can be found that states that the integral $\sum\int\int dq^i dp_i$ (the integrals to be evaluated over any arbitrary two-dimensional surface in phase space) is invariant under canonical transformations (Theorem of Poincaré). It is shown by a simple example that this is not true. A few sources of the mistake are discussed and a correct formulation of the theorem is given. *J. Haantjes* (Leiden).

Klein, Joseph. Sur les trajectoires d'un système dynamique dans un espace finslérien ou variationnel généralisé. C. R. Acad. Sci. Paris 238, 2144–2146 (1954).

The equations of motions of a nonconservative holonomic dynamical system S can be given in the form:

$$\frac{d}{du}\partial_{\dot\alpha}\mathfrak{L}-\partial_\alpha\mathfrak{L}=M_\alpha;\quad \alpha=1,\cdots,n+1;\quad x^{n+1}=t,$$

where $\mathfrak{L}$ and M_α are homogeneous of degree 1 in $\dot x^\alpha=dx^\alpha/du$ and $M_\alpha\dot x^\alpha=0$. Introducing the tensor $S_{\alpha\beta}=\frac{1}{2}(\partial_{\dot\beta}M_\alpha-\partial_{\dot\alpha}M_\beta)$, it is shown that the equations of motion are given by the system associated with the quadratic form

$$\Omega=d\partial_{\dot\alpha}\mathfrak{L}\wedge dx^\alpha+\tfrac{1}{2}S_{\alpha\beta}dx^\alpha\wedge dx^\beta.$$

The condition is given in order that the paths of a system S be the geodesics of a variational space connected with the function $\mathfrak{L}$. *J. Haantjes* (Leiden).

*Kähler, Erich. **Osservazioni a proposito della dinamica.** Convegno Internazionale di Geometria Differenziale, Italia, 1953, pp. 82–98. Edizioni Cremonese, Roma, 1954. 4000 Lire.

Following an idea of Riemann about the dynamical origin of the metric spaces, which makes it possible to introduce a dynamical terminology in geometry, the author firstly extends the dynamical terminology to algebra. Then a number of postulates are introduced in order to make it possible according to the author's expectations to attack the problems of all organic phenomena by algebraic methods. The theory is illustrated by a few physical examples. *J. Haantjes* (Leiden).

Garnier, Maurice. La balistique extérieure moderne en France. Mém. Artillerie Française 28, 117–234 (1954).

This is an authoritative exposition of exterior ballistic methods in France, with a review of the progress made since Bernoulli and Euler. In World War I, under urging from Charbonnier, the author with Haag and Marcus introduced the G-H-M-1917 method of short arcs. Later advances have consisted chiefly of revisions in the formulas for air density aloft, the use of Mach numbers, adoption of curvilinear coordinates, with local geodesic altitude and mean sea level as zero elevation, acceptance of variation of gravitational acceleration with altitude. This exposition covers the most recent revision, the G-H-M-1953 method. Computation sheets are laid out with full explanations and an example carried through. Despite a brief reference to the gyroscope, this exposition treats the projectile as a particle without drift, and unaffected by questions of stability. The short arcs are handled separately, and the errors introduced by simplifying assumptions as, for example, of the Siacci-type are examined. Five-figure logarithms are used. However the tables needed for air density and retardation coefficient are not exhibited here. Two "adjustment coefficients" are used, the first is a form-factor for the projectile, the second selects a retardation coefficient from a fixed one-parameter family of such functions. No further freedom in the determination of air resistance is provided for. Happily, modern American ballistic practice is free from computational restraints (of the sort here treated) and from one-parameter retardation laws, and has long implemented its recognition of stability coefficients especially in air-to-air fire. *A. A. Bennett* (Providence, R. I.).

Hydrodynamics, Aerodynamics, Acoustics

*Milne-Thomson, L. M. **Le théorème complexe de Stokes.** Mémoires sur la mécanique des fluides offerts à M. Dimitri P. Riabouchinsky, pp. 233–239. Publ. Sci. Tech. Ministère de l'Air, Paris, 1954. 3000 francs.

The complex form of Stokes's theorem was first presented in the author's book, "Theoretical hydrodynamics" [2nd ed., Macmillan, New York, 1949, p. 130; these Rev. 11, 471]. This paper reviews the derivation of the formula, and gives several simple applications of it. *J. B. Serrin.*

Pyhteev, G. N. Determination of the two-dimensional potential motion of an incompressible fluid from given values of the direction of its velocity. Akad. Nauk SSSR. Prikl. Mat. Meh. 18, 379–380 (1954). (Russian)

Let $Ve^{i\theta}$ be the velocity. Since θ and $-\ln V$ are conjugate harmonic functions, the velocity potential corresponding to given harmonics θ is readily found by quadratures. *J. H. Giese* (Ann Arbor, Mich.).

Lombardo, Giorgio. Sulla deduzione dei principi di reciprocità nell'idraulica e nella termodinamica. Atti Accad. Ligure 9 (1952), 332–334 (1953).

If $\mathbf{q}=-\mu$ grad h, div $\mathbf{q}=0$, $\mathbf{q}'=-\mu$ grad h', div $\mathbf{q}'=0$, the application of Gauss's theorem to a closed surface σ gives the reciprocal theorem

$$\int_\sigma h'\mathbf{q}\times\mathbf{n}d\sigma=\int_\sigma h\mathbf{q}'\times\mathbf{n}d\sigma,$$

where $\times$ denotes scalar multiplication. *L. M. Milne-Thomson* (Greenwich).

*Truesdell, C. **Le pendule hydraulique.** Mémoires sur la mécanique des fluides offerts à M. Dimitri P. Riabouchinsky, pp. 383–396. Publ. Sci. Tech. Ministère de l'Air, Paris, 1954. 3000 francs.

This is an elementary discussion, neglecting surface tension, of the pressure inside a drop of perfect incompressible fluid inside a toroidal tube in a uniform gravitational field. *E. Pinney* (Berkeley, Calif.).

Ghosh, N. L. A theory of resistance in potential flows. I–IV. Proc. Nat. Inst. Sci. India 20, 74–103 (1954).

According to the author's abstract, this paper "explains that the age old paradoxes of the classical theory [of fluid resistance] really arise out of neglect of viscosity coupled with the hypothesis of no slip on the boundary", and "points out that a change of outlook . . . brings in a remarkable simplification into the whole problem." His "change of outlook" is, essentially, that resistance (drag) in a viscous fluid should be computed by means of the forces exerted by the fluid on the outside edge of the boundary layer. It is perhaps superfluous to remark that this neglects any drag resulting from dissipation in the boundary layer or wake, and, in particular, leads to zero drag for a flat plate parallel to a uniform stream. *J. B. Serrin* (Minneapolis, Minn.).

Tan, H. S. A unique law for ideal incompressible flow with preserved pattern of finite separation. Quart. Appl. Math. 12, 78–80 (1954).

En étudiant des écoulements non stationnaires d'un fluide parfait et incompressible autour d'un coin ou d'un obstacle plane, von Kármán et Prandtl sont arrivés tous deux à la loi de la vitesse $U(t)=U_0/(1-a_0t\dot U_0^{-1})$. L'auteur montre que cette loi est générale et unique pour les schémas en-

visagés, lorsqu'on suppose que le mouvement est de la forme $U(t)\phi(x, y)$. *R. Gerber* (Toulon).

Garabedian, P. R., McLeod, Edward, Jr., and Vitousek, Martin. Recent advances at Stanford in the application of conformal mapping to hydrodynamics. Proceedings of the symposium on special topics in applied mathematics, Northwestern University, 1953 Amer. Math. Monthly **61**, no. 7, part II, 8–10 (1954).

Bloh, È. L. The horizontal hydrodynamic impact of a sphere in the presence of a free surface of the fluid. Akad. Nauk SSSR. Prikl. Mat. Meh. **17**, 579–592 (1953). (Russian)

The author considers a sphere S, half immersed in an ideal fluid which fills a lower half-space and which is initially at rest. A prescribed initial velocity in the horizontal plane is then imparted to S by means of an impulsive force. Using an analytic continuation across the free surface of the fluid, introduced by Sedov [Trudy Central. Aero-Gidrodinam. Inst. no. 187, 1–28 (1934)], the author reduces the problem of determining the velocity distribution in the fluid at the moment of impact to an exterior Neumann problem for the impulsive pressure. The solution of this problem is given explicitly as an absolutely and uniformly convergent series of associated Legendre functions of the first kind and first order.

By the same method the author determines, (a) the velocity distribution in a fluid which fills a hemispherical bowl when the bowl is subjected to a horizontal impulsive force, and (b) the velocity distribution in a fluid which, together with a concentric floating sphere, fills a hemispherical bowl. In this last case the sphere and the bowl are subjected to distinct but simultaneous horizontal impulses.
R. Finn (Los Angeles, Calif.).

Bloh, È. L. Horizontal impact of an ellipsoid of rotation on an ideal fluid in the presence of a free surface. Akad. Nauk SSSR. Prikl. Mat. Meh. **17**, 705–726 (1953). (Russian)

The author extends the results cited in the preceding review, using the same method, to the case of an ellipsoid of revolution half submerged in an ideal fluid which fills a lower half-space. It is assumed that one of the principal axes of the ellipsoid is orthogonal to the free surface of the fluid. The various cases are distinguished, and the solutions are given explicitly as series of associated Legendre functions of the first and second kind. The author considers also the case of an ellipsoidal bowl half filled with fluid.
R. Finn (Los Angeles, Calif.).

Rubin, Hanan. The dock of finite extent. Comm. Pure Appl. Math. **7**, 317–344 (1954).

Le problème considéré est celui du dock de longueur finie en eau profonde. L'auteur est parvenu à établir pour ce problème l'existence d'une solution ayant l'allure d'une houle incidente progressive avec une phase, une amplitude et une longueur d'onde, fixées arbitrairement. De plus, on se donne l'importance relative du déferlement (ou encore de la singularité logarithmique) sur l'arête du dock et les résultats obtenus mettent en évidence l'influence de cette grandeur sur le coefficient de réflexion de l'onde incidente.

Le mouvement étant supposé petit et irrotationnel, on sait que l'on est conduit à résoudre un problème harmonique dans un demi-plan, avec des conditions aux limites linéaires, et le théorème d'existence est obtenu par des méthodes variationnelles. Ces méthodes ne s'appliquant pas directement au problème posé pour le potentiel des vitesses $\phi(x, y)$, l'auteur est parvenu à tourner cette difficulté en considérant un problème auxiliaire du même type, portant sur la fonction $\psi=\phi_y-\phi$, déjà introduite avec succès dans ce genre de questions par Stoker [Quart. Appl. Math. **5**, 1–54 (1947); ces Rev. **9**, 163]. Il est d'abord montré qu'à partir d'une solution de ce deuxième problème, on peut construire le potentiel ϕ d'un mouvement satisfaisant à toutes les conditions imposées.

Le mouvement doit nécessairement comporter une singularité sur l'arête du dock, et cette singularité est introduite en prenant pour ψ une expression de la forme $\psi^R+\psi^*$ où ψ^R est la fonction correspondante du problème du dock semi-infini, construite explicitement par Friedrichs et Lewy [Communications on Appl. Math. **1**, 135–148 (1948); ces Rev. **10**, 336], et où ψ^*, qui est la nouvelle inconnue, est supposée régulière. Le problème aux limites posé pour ψ^* est alors ramené à un problème de minimum pour une certaine fonctionnelle de ψ^*, problème qui admet une solution comme le démontre l'auteur. *R. Gerber* (Toulon).

Müller, W. Die Bewegung eines Rotationskörpers in der reibungslosen Flüssigkeit und das instabile Moment der Druckkräfte. Österreich. Ing.-Arch. **8**, 171–184 (1954).

When an elongated solid of revolution moves with velocity U at incidence β to its longitudinal axis, the fluid exerts a moment (in general tending to increase β). The author exhibits this moment in the form

$$\tfrac{1}{2}\pi\rho U^2 \sin 2\beta\left[\int \eta F_2 dx-2\int \eta F_1 d\eta\right]+\tfrac{1}{2}\pi\rho\dot{U}\sin\beta\int F_2\eta d(r^2),$$

where η, r are distances respectively from the axis and the origin, and the velocity potential is

$$U\cos\beta F_1+U\sin\beta F_2\cos\varphi.$$

The author determines F_1 and F_2 for a prolate spheroid and evaluates the moment and the virtual mass. It seems to the reviewer that the treatment could be much simplified by the direct use of Kirchhoff's equations.
L. M. Milne-Thomson (Greenwich).

Seth, B. R. Generalized singular points with applications to flow problems. Proc. Indian Acad. Sci. Sect. A. **40**, 25–36 (1954).

Generalized singular points are used to discuss irrotational and viscous flows produced in an infinite liquid by a moving solid. It is found that the irrotational motion of translation is the same as that due to a generalized doublet and that of rotation the same as that due to a "rotation singular point". The corresponding viscous problems are solved by superposing on the irrotational motion a solution due to a concentrated force or a couple. (From the author's summary.)

The reviewer feels obliged to add that this paper is written in an extremely vague style and that he is unable to verify the stated results. *J. B. Serrin.*

✶Darrieus, G. Entretien, par extension suivant son axe, d'une trombe rectiligne dans un fluide visqueux. Mémoires sur la mécanique des fluides offerts à M. Dimitri P. Riabouchinsky, pp. 41–47. Publ. Sci. Tech. Ministère de l'Air, Paris, 1954. 3000 francs.

To demonstrate that a rectilinear vortex parallel to, and in front of, an obstacle with a plane surface, placed in a steady oncoming stream, can be maintained by viscosity,

the author considers a special problem in which the flow is assumed to be a linear vortex-source combination in a viscous incompressible fluid, superposed on a flow parallel to the plane of the obstacle with a velocity varying linearly with distance, measured from the plane of symmetry, and in which the effect of solid boundary is neglected. Accordingly, it is concluded that the circulation around the vortex is invariant with time and decays in the radial direction with a rate proportional to an exponential factor.

Y. H. Kuo (Ithaca, N. Y.).

Collatz, Lothar, und Görtler, Henry. Rohrströmung mit schwachem Drall. Z. Angew. Math. Physik **5**, 95–110 (1954).

The problem considered is that of steady laminar flow at high Reynolds number in a straight circular tube. It is supposed that there is an axially-symmetrical swirl but that the rotational velocity is small compared with the axial velocity. The corresponding eigen-value problem is examined and the first five eigen-values and the corresponding eigen-functions are obtained by numerical methods. An example is taken in which the flow in the initial cross-section consists of a rigid rotation superimposed upon the Poiseuille flow; the decay of this motion downstream is calculated.

D. C. Pack (Glasgow).

Batchelor, G. K. The skin friction on infinite cylinders moving parallel to their length. Quart. J. Mech. Appl. Math. **7**, 179–192 (1954).

The author calculates the flow produced by an infinite cylinder starting impulsively to move along its axis. It is shown that when the time t is small, the first approximation gives a force which is the same as that for a flat plate of infinite width. The second approximation takes the shape of the cylinder into account and the force on unit length of cylinder is determined approximately in terms of the number of corners and their angles in the cylinder cross-section. If there are no corners, the force on unit length of the cylinder is the same, to this approximation, as that on a circular cylinder of the same perimeter. For large t, the determination of force is reduced to a potential problem for which it is found that the shape of the cylinder has a diminishing effect on the velocity distribution as a whole; and therefore the frictional force can be related to that on a circular cylinder.

Y. H. Kuo (Ithaca, N. Y.).

***Hogner, Einar. L'influence des bords sur les couches limites.** Mémoires sur la mécanique des fluides offerts à M. Dimitri P. Riabouchinsky, pp. 129–134. Publ. Sci. Tech. Ministère de l'Air, Paris, 1954. 3000 francs.

By Oseen's approximation, it is shown that the local friction coefficient of a flat plate of width $2b$ and of length $2l$ satisfies an integral equation of the Volterra type. With the semi-infinite plate as initial approximation, the integral equation is solved by iteration. The expression for the first approximation is obtained.

Y. H. Kuo (Ithaca, N. Y.).

Wilkinson, J. Some examples of three-dimensional effects in boundary layer flow. Aeronaut. Quart. **5**, 73–84 (1954).

This is a study of a type of three-dimensional laminar boundary-layer flow that is nearly two-dimensional, the potential velocities being $U=U(\xi)$ and $V=\alpha\eta$, where ξ, η are orthogonal coordinates and α is a small parameter. The components U and V are called "main flow" and "cross flow," respectively. It is shown that to the first approximation the boundary-layer velocity components u and w are independent of α; i.e., the "independence principle" applies [as in a somewhat similar problem of Fogarty, J. Aeronaut. Sci. **18**, 247–252 (1951); these Rev. **14**, 328]. The present author, however, goes beyond the first approximation and determines the effect of the cross flow on u and w; i.e., he calculates the terms of order α in these quantities. The cases studied are (1) $U=\xi^m$, for several values of m, and (2) $U=1-\xi$. The first is handled by solution of the differential equations and also, for comparison, by a Pohlhausen procedure. The second is treated by the Pohlhausen procedure only. In both cases it appears that the existence of the cross flow tends to increase the skin friction and to delay separation of the main flow. Doubt is cast on the accuracy of the latter conclusion because of known inadequacies of the methods near separation, but it is suggested that the general tendencies are correct.

W. R. Sears (Ithaca, N. Y.).

Wieghardt, K. E. G. On a simple method for calculating laminar boundary layers. Aeronaut. Quart. **5**, 25–38 (1954).

In calculating approximately laminar boundary layers in axi-symmetric and plane flows, the author proposes to modify and extend the one-parameter (shape) method of Walz [Ing.-Arch. **16**, 243–248 (1948); these Rev. **10**, 337], based on the momentum and energy equations. To simplify the calculations, it is assumed that for both axi-symmetric and plane flows, the coefficients appearing in the total differential equations are universal functions of this shape parameter, and therefore are determined by the special Falkner-Skan velocity profiles. In the plane case with linear pressure-gradient variation, a comparison with previous exact as well as approximate calculations has been made. For axi-symmetric boundary layers, three different cases: (a) a half-body, (b) hemisphere and cylinder and (c) ½-calibre rounded head and cylinder have been calculated as examples.

Y. H. Kuo (Ithaca, N. Y.).

Bellman, Richard, and Pennington, Ralph H. Effects of surface tension and viscosity on Taylor instability. Quart. Appl. Math. **12**, 151–162 (1954).

On considère deux fluides pesants, de densités différentes, dont la surface de séparation présente à l'état initial une perturbation de forme sinusoïdale ayant une amplitude petite par rapport à la longueur d'onde. On suppose que le système est soumis, outre les forces de pesanteur, à une accélération verticale et on se propose d'en étudier la stabilité. Ce problème a été traité par G. I. Taylor [Proc. Roy. Soc. London. Ser. A. **201**, 192–196 (1950); ces Rev. **12**, 58] lorsque les fluides sont supposés parfaits et les auteurs étudient l'influence de la viscosité et de la tension superficielle sur la stabilité du schéma précédent.

R. Gerber (Toulon).

***Lin, C. C. Hydrodynamic stability.** Proceedings of Symposia in Applied Mathematics, Vol. V, Wave motion and vibration theory, pp. 1–18. McGraw-Hill Book Company, Inc., New York-Toronto-London, 1954. $7.00. Hav

The author defines hydrodynamic stability for the Navier-Stokes equations in terms of linearized differential equations, obtaining characteristic boundary value problems. The results are compared with the known exact solutions of Couette and Poiseuille flows, and stability criteria are derived for other particular flows. A discussion of the asymptotic expansions of the solutions of these stability

equations is included, referring principally to W. Wasow [Ann. of Math. (2) 58, 222–252 (1953); these Rev. 15, 874]. Further discussion pertains to the limiting cases of the Reynolds number tending to infinity, and the viscosity to zero, noting, in particular, that situations arise in which the viscosity cannot be neglected, however small.

G. Latta (Stanford, Calif.).

*Chandrasekhar, S. **Examples of the instability of fluid motion in the presence of a magnetic field.** Proceedings of Symposia in Applied Mathematics, Vol. V, Wave motion and vibration theory, pp. 19–27. McGraw-Hill Book Company, Inc., New York-Toronto-London, 1954. $7.00.

Thermal and rotational instability of solutions of the Navier-Stokes equations under the influence of a magnetic field are discussed, and the dependence on the magnetic field is demonstrated by a table of the critical Rayleigh and Taylor numbers. The characteristic boundary-value problems for the stability criteria are solved by means of a variational reformulation of the problems. The results are of such a nature as to admit of experimental verification.

G. Latta (Stanford, Calif.).

Chandrasekhar, S. Problems of stability in hydrodynamics and hydromagnetics. Monthly Not. Roy. Astr. Soc. 113 (1953), 667–678 (1954).

George Darwin lecture, delivered 13 November 1953.

Chandrasekhar, S. The stability of viscous flow between rotating cylinders. Mathematika 1, 5–13 (1954).

For a viscous flow between rotating concentric cylinders ($R_1<r<R_2$) the equations of hydrodynamics admit the stationary solution $V=Ar+B/r$, where

$$A=\Omega_1(1-\mu R_2^2/R_1^2)/(1-R_2^2/R_1^2),$$
$$B=\Omega_1 R_1^2(1-\mu)/(1-R_1^2/R_2^2), \quad \mu=\Omega_2/\Omega_1.$$

The equations governing the marginal stability of a symmetric disturbance of wave number λ in the axial direction may be written as $(D^2-a^2)^3v=-a^2T(1+\alpha\zeta)v$, with $v=(D^2-a^2)v=D(D^2-a^2)v=0$ for $\zeta=0$ and $\zeta=1$. Here

$$\zeta=(r-R_1)/d, \quad d=(R_2-R_1), \quad a=\lambda d,$$
$$\alpha=-(1-\mu), \quad T=-4A\Omega_1 d^4/\nu, \quad D=d/d\zeta,$$

and ν is the kinematic viscosity. In deriving this equation, it has been assumed that d is small and also $A+B/r^2$ has been approximated by a linear profile. Of interest is the minimum value of T, T_c, over all positive a for given μ.

For $\mu\to 1$, $T_c\to 3416/(1+\mu)$ [Taylor, Philos. Trans. Roy. Soc. London. Ser. A. 223, 289–343 (1923)]. For negative μ, $1+\alpha\zeta$ vanishes in $0\leqq\zeta\leqq 1$ and hence the boundary-value problem becomes singular. For $\mu\to-\infty$, Meksyn obtained $T_c\to C(1-\mu)^4$, where $C=1130$ [Proc. Roy. Soc. London. Ser. A. 187, 480–491, 492–504 (1946); these Rev. 8, 415, 382]. In this paper T_c and a_c are computed for the range $-2<\mu<1$, by a method analogous to the Galerkin method. The differential equation is written as the pair of equations $(D^2-a^2)^2W=(1+\alpha\zeta)\psi$, $(D^2-a^2)\psi=-a^2TW$ where $W=(D^2-a^2)v$. The function ψ is expanded in a sine series, W is computed from the first equation, and the second then gives an equation for T. For values of $\mu>-1$ three terms of the sine series for ψ were used, for $\mu<-1$ four terms were used to insure sufficient accuracy. For $\mu\to-\infty$, $T_c\to 1180(1-\mu)^4$, $a(T_c)\to 2.03(1-\mu)$. A table of values of T_c and $a(T_c)$ is given for $-2<\mu<1$.

R. C. Di Prima.

Pai, S. I. On a generalization of Synge's criterion for sufficient stability of plane parallel flows. Quart. Appl. Math. 12, 203–206 (1954).

Les conditions de stabilité obtenues par Synge [Amer. Math. Soc. Semicentennial Publ., v. 2, New York, 1938] pour les mouvements à la Poiseuille et à la Couette d'un fluide incompressible sont généralisées aux écoulements présentant des points d'inflexion et ayant une frontière rejetée à l'infini.

R. Gerber (Toulon).

Baatard, François. Mouvement de rotation différenciée permanente d'un fluide. Bull. Tech. Suisse Romande 80, 237–240 (1954).

L'auteur rappelle les origines diverses des mécaniques statistiques, et l'intérêt de la théorie moderne des fonctions aléatoires pour nombre d'applications physiques. Il explique en quoi la vitesse d'un fluide turbulent tel que le vent atmosphérique doit être considéré, suivant G. Dedebant et Ph. Wehrle, comme un vecteur aléatoire entre les composantes duquel existent des corrélations. Il développe ensuite un exemple schématique dans lequel, à un vecteur aléatoire dont les composantes sont deux oscillateurs simples aléatoires, de même période donnée, on associe le mouvement d'un fluide à deux dimensions. Moyennant quelques hypothèses, ce mouvement est une rotation différenciée, pouvant dégénérer en rotation solide ou en repos.

Le rapporteur désire ajouter que cet exemple a été entièrement traité par lui même, non dans l'ouvrage de 1949 cité en référence, mais dans Groupement Français Dévelop. Rech. Aéronaut. Rap. Tech. no. 28 (1946) [ces Rev. 11, 699].

J. Bass (Chaville).

Lighthill, M. J. Mathematical methods in compressible flow theory. Comm. Pure Appl. Math. 7, 1–10 (1954).

Bers, Lipman. Results and conjectures in the mathematical theory of subsonic and transonic gas flows. Comm. Pure Appl. Math. 7, 79–104 (1954).

Cet article nous offre une remarquable étude de synthèse sur la théorie mathématique des profils placés dans un écoulement stationnaire subsonique ou transsonique. Après avoir rappelé les approches classiques et leurs insuffisances, l'auteur analyse l'orientation des travaux consacrés à ces questions depuis quelques années. Si, du point de vue mathématique, il a été possible de prouver l'existence et l'unicité de l'écoulement lorsque celui ci est toujours subsonique, il n'en est plus de même lorsque la vitesses sonique est atteinte dans l'écoulement. L'une des principales difficultés que l'on rencontre dans ce cas est que l'équation aux dérivées partielles n'est plus d'un type invariable dans le domaine où est définie la solution; on a affaire à une équation du type mixte. L'auteur passe en revue les travaux consacrés à ces équations du type mixte; à la lumière des résultats déja acquis il formule trois énoncés non encore démontrés mais très vraisemblables sur la non-existence d'un écoulement transsonique continu autour d'un profil. L'article est suivi d'une bibliographie très précieuse.

P. Germain (Paris).

von Mises, R. Discussion on transonic flow. Comm. Pure Appl. Math. 7, 145–148 (1954).

Sagomonyan, A. Ya. The method of characteristics for the unsteady axisymmetric self-similar motion of a fluid. Vestnik Moskov. Univ. Ser. Fiz.-Mat. Estest. Nauk 8, no. 12, 63–68 (1953). (Russian)

An axisymmetric flow field is self-similar ("auto-model") if the velocity components, pressure, density, and entropy

are functions of x/l and y/l only, where x, y are cylindrical coordinates. The author sketches methods to be used in three typical boundary problems which may occur in constructing numerical solutions of the characteristic equations for either linearized or non-linearized isentropic flow.

J. H. Giese (Ann Arbor, Mich.).

Müller, Werner. Hodographenmethode der Gasdynamik bei quadratisch approximierter Adiabate. S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. **1953**, 313–330 (1954).

The stream function ψ of an irrotational flow in the $z=x+iy$ plane satisfies (i) $\psi_{\omega\omega}+\psi_{\theta\theta}+\psi_\omega d\ln \mathfrak{L}(\omega)/d\omega=0$ where $we^{i\theta}$ is the velocity,

$$d\omega=(1-M^2)^{\frac{1}{2}}dw/w, \quad \text{and} \quad \mathfrak{L}(\omega)=(1-M^2)^{\frac{1}{2}}\rho_1/\rho.$$

Sauer [same S.-B. **1951**, 65–71 (1952); these Rev. **14**, 598] has determined the explicit form of pressure density relation such that (i) has solutions $\psi^*(\omega, \theta)/\omega$ where

$$\partial^2\psi^*/\partial\omega^2+\partial^2\psi^*/\partial\theta^2=0.$$

At any point (p, ρ) the isentropic relation $p/p_0=(\rho/\rho_0)^\gamma$ can be approximated correctly to terms of second order in $1/\rho$ by one of Sauer's adiabats, but only for a limited subsonic interval of values of $w\neq 0$. Now let $H(\overline{\Omega})=\varphi^*+i\psi^*$ be an analytic function of $\overline{\Omega}=e^{\omega-i\theta}/c_1$ for some constant c_1. Then the Molenbroek transformation becomes

$$Cdz=dH/\overline{\Omega}-\Omega d\bar{H}-d[(H-\bar{H})(\overline{\Omega}-1/\Omega)/\ln (\Omega\overline{\Omega}c_1^2)]$$

for another constant C. The author shows that for such flows with circulation about airfoils

$$dH/d\overline{\Omega}=b_{-2}/(\overline{\Omega}-\eta_\infty)^2+b_{-1}/(\Omega-\eta_\infty)+\text{regular terms},$$

where $\overline{\Omega}=\eta_\infty$ at $z=\infty$, and b_{-1} is pure imaginary. These flows can be related to incompressible flows with circulation about airfoils in a $Z=X+iY$ plane by setting

$$\overline{\Omega}=G(Z)=\eta_\infty(1+\kappa_1/Z+\kappa_2/Z^2+\cdots)$$

for some analytic $G(Z)$ and taking as velocity potential $H(G(Z))$. If $\overline{\mathfrak{W}}=dH/dZ$, then, in

$$k(Z)=\overline{\Omega}/\overline{\mathfrak{W}}=(\eta_\infty/W_\infty)(1+\beta_1/Z+\beta_2/Z^2+\cdots),$$

β_1 is proportional to the circulation about the profile $\psi^*=0$ in the Z-plane. Conditions on $H(Z)$, $k(Z)$, etc. are listed under which this procedure can be inverted, and as an example incompressible flow with circulation about a circular arc is transformed into compressible flow with circulation and in accordance with a Sauer adiabat about a cambered airfoil bounded by two approximately circular arcs.

J. H. Giese (Ann Arbor, Mich.).

***Kline, S. J., and Shapiro, A. H. One-dimensional, steady gasdynamics for an arbitrary fluid.** Mémoires sur la mécanique des fluides offerts à M. Dimitri P. Riabouchinsky, pp. 171–202. Publ. Sci. Tech. Ministère de l'Air, Paris, 1954. 3000 francs.

Dans cet article les auteurs étudient certaines propriétés des écoulements unidimensionnels sans frottement ou avec frottement, pour des lois d'état du fluide extrêmement générales. Ce fluide est simplement supposé être une substance pure à une seule phase, c'est à dire que toutes les grandeurs thermodynamiques relatives au fluide sont des fonctions connues de deux d'entre elles prises comme variables indépendantes. Les résultats obtenus ne sont pas tous essentiellement nouveaux, mais une telle présentation générale est loin d'être inutile. La conclusion de cet article est que d'une façon générale les fluides usuels se comportent comme les gaz parfaits; certaines anomalies peuvent toutefois se produire; c'est contre elles que les auteurs nous mettent en garde.

P. Germain (Paris).

Chester, W. Unsteady compressible flow in ducts. Quart. J. Mech. Appl. Math. **7**, 247–256 (1954).

Let a one-dimensional rarefaction wave be produced in a channel by impulsive retraction of a piston at the escape speed of the fluid. Suppose that along a finite length (transition section) of the channel there is a slight variation of width. To determine effects of this deformation, the author adds to the velocity potential of the rarefaction wave a perturbation potential $\varphi(x, y, t)$, where the x-axis (y-axis) is parallel (normal) to the axis of the channel, and linearizes the partial differential equation satisfied by φ. By operational methods φ is expressed as a Fourier cosine series in terms of the variable y. Consideration of the first term of the series suggests that beyond the transition section the perturbation velocity at a point moving with the wave tends to decrease the flow velocity by an amount inversely proportional to time, while at a fixed point in the channel the perturbation velocity increases linearly with time. In the transition section disturbances accumulate and tend to increase the flow velocity if there is a throat, eventually producing a sonic regime. On this basis an attempt is made to describe the establishment of steady subsonic-supersonic flow in a converging-diverging nozzle.

J. H. Giese.

Kaplan, Carl. The small-disturbance method for flow of a compressible fluid with velocity potential and stream function as independent variables. NACA Tech. Note no. 3229, 18 pp. (1954).

Two-dimensional compressible flow problems are attacked, using the position variables, x and y, as dependent variables, and using the potential function ϕ and the stream function ψ as independent variables. An iteration procedure is used in which x and y are expanded in powers of a small parameter, the coefficients being functions of ϕ and ψ. This procedure has two advantages: first, the iteration equations are of the first order instead of the second; and second, boundary conditions are usually given in terms of streamlines $\psi=\text{const.}$, so the application of boundary conditions is easier than is usually the case in nonlinear problems. The iteration equations further simplify when the complex variables $z=x+iy$, $w=\phi+i\psi$ are introduced. Their integration then becomes straightforward.

The theory is applied to the study of subsonic flow along a wavy wall, which the author has studied extensively in the past. Although many details of the calculations are necessarily suppressed, it seems clear that the present method offers advantages in computational economy.

E. Pinney (Berkeley, Calif.).

Keller, Joseph B. Multiple shock reflection in corners. J. Appl. Phys. **25**, 588–590 (1954).

In this article it is pointed out that there are certain special cases of a finite shock entering a corner from the concave side in which no diffraction occurs. In these cases it is possible to solve the entire problem by using the theory of regular reflection, which yields an explicit solution by algebraic means alone. (From the author's summary.)

D. C. Pack (Glasgow).

★A selection of graphs for use in calculations of compressible airflow. Prepared on behalf of the Aeronautical Research Council by the Compressible Flow Tables Panel: L. Rosenhead, W. G. Bickley, C. W. Jones, L. F. Nicholson, H. H. Pearcey, C. K. Thornhill, and R. C. Tomlinson. Oxford, at the Clarendon Press, 1954. x+115 pp. $13.45.

These graphs are a selection from the same panel's tables [1952; these Rev. **14**, 510] and other sources. They give the most frequently used data in four categories; isentropic flow, normal shocks, oblique shocks, and conical flow, together with graphs for computing Reynolds numbers. The ratio of specific heats has been taken equal to 1.4. The range of Mach numbers is 0 to 5. The pages are large (11 by 15 inches) and the graphs are exceptionally clear and easy to read. In every case there is a brief compilation of the equations used, with references. There are numerous diagrams that define the symbols and assist the user of the charts. *W. R. Sears* (Ithaca, N. Y.).

★Birkhoff, Garrett, and Walsh, John M. Conical, axially symmetric flows. Mémoires sur la mécanique des fluides offerts à M. Dimitri P. Riabouchinsky, pp. 1–12. Publ. Sci. Tech. Ministère de l'Air, Paris, 1954. 3000 francs.

This is an investigation of conical axially symmetrical irrotational flows of a compressible fluid in which the pressure is an increasing function of the density. The flow is called relatively subsonic or relatively supersonic at a point P according as the radial line through P is or is not included in the Mach cone which emanates from P. It is then shown that a typical streamline in any flow of the type under consideration is initially supersonic and parallel to the axis. It then becomes concave inwards, passes through a point of inflexion, becomes concave outwards, and is deflected by a shock into relatively subsonic divergent flow. Far downstream the flow becomes parallel to a radial line. The entire family of such flows is three-parametric, the two-parametric families of Taylor-Maccoll and of Busemann are limiting cases. *A. Robinson* (Toronto, Ont.).

Lance, G. N. The delta wing in a non-uniform supersonic stream. Aeronaut. Quart. **5**, 55–72 (1954).

In this paper, the analysis of cone fields of higher order in linearised supersonic flow is developed in close analogy with the well-known theory of cone fields of order one [cf., e.g., S. Goldstein and G. N. Ward, Aeronaut. Quart. **2**, 39–84 (1950); these Rev. **15**, 177]. An elegant set of compatibility conditions is obtained by means of Euler's theorem on homogeneous functions. The analysis is applied to the flow round a Delta wing with subsonic leading edges for a downwash distribution which is a quadratic function of the coordinates. The author states that he has checked his results by means of the method of hyperboloido-conal coordinates. It may be mentioned that the same method has been applied to this class of problems by G. M. Roper [Quart. J. Mech. Appl. Math. **1**, 327–343 (1948); these Rev. **10**, 274]. The results are interpreted in terms of a wing which is immersed in a non-uniform supersonic stream (e.g. a tail plane in the wake of the main plane).

A. Robinson (Toronto, Ont.).

Nickel, K. Über spezielle Systeme von Tragflügelgittern. I. Theorie der tragenden Linie. Ing.-Arch. **22**, 108–120 (1954).

This is a study of the aerodynamics of infinite arrays of wings of finite span. The author first treats a single cascade of identical, equally spaced wings, whose trailing-vortex sheets occupy the positions $z=nh_2$, $a+nh_1\leqq y\leqq b+nh_1$, $n=\cdots,-2,-1,0,1,2,\cdots$, where a, b, h_1, and h_2 are constants and the x-axis is the stream direction. The induced velocity at one of these sheets in the Trefftz plane is calculated as an integral over the circulation distribution $\Gamma(y)$. The integral equation is inverted and the total lift is calculated, but in general these formulas are not useful because both components (downwash and sidewash) are involved while only downwash will be known. For the special cases $h_1=0$ and $h_2=0$, which represent, respectively, a vertical cascade and an infinite row of wings beside one another, the sidewash is not needed. For these two cases, the distributions $\Gamma(y)$ for minimum induced drag are evaluated.

The author proceeds to treat arrays consisting of a finite number m of such cascades, arranged parallel to one another but not necessarily having the same values of h_1 and h_2. (Of course, h_1/h_2 must be the same for each cascade.) The integral equation for this case is inverted; the result involves m arbitrary constants, which, the author states, will be determined uniquely by the condition that Γ must be zero at every wing tip. Finally he considers a similar array of cascades, in each of which the wings have alternately the circulation distributions Γ and $-\Gamma$. *W. R. Sears.*

Krasil'ščikova, E. A. Unsteady motions of a wing of infinite span. Izvestiya Akad. Nauk SSSR. Otd. Tehn. Nauk **1954**, no. 2, 25–41 (1954). (Russian)

This gives details of work reported earlier [Doklady Akad. Nauk SSSR (N.S.) **94**, 397–400 (1954); these Rev. **15**, 910]. The author applies to linearized arbitrary unsteady two-dimensional flows techniques for solving the two-dimensional wave equation that she has previously used for three-dimensional wings in linearized steady or harmonically oscillating flow [Moskov. Gos. Univ. Učenye Zapiski **154**, Mehanika **4**, 181–239 (1951); these Rev. **14**, 815]. The velocity potential $\varphi(t,x,z)$ is obtained by superposition of sources of strength $\varphi_z(t,x,0)$ on $z=0$, where $\partial\varphi/\partial z=\varphi_z(t,x,z)$. In regions of $z=0$, where the intensity $\varphi_z(t,x,0)$ is not known a priori for a prescribed airfoil motion, it is found by solving a sequence of integral equations. The examples considered illustrate procedures for treating airfoil motions that start from rest and are accelerated to sub- or supersonic speeds, or are accelerated from long-established supersonic motion to subsonic speed. Also considered is the possibility of deforming the airfoil, as by extension or retraction of a flap. *J. H. Giese* (Ann Arbor, Mich.).

Krzywoblocki, M. Z. V. On the general theory of downwash behind a finite wing in supersonic range. Bull. Calcutta Math. Soc. **45**, 21–40 (1953).

The author develops a supersonic aerofoil theory which is based in the first instance on the exact equation of potential compressible flow. It is suggested that this equation can be solved for given boundary conditions at the aerofoil by applying Picard's iteration method to Green's formula, regarded as an integro-differential equation for the velocity potential. The calculation of the vortex wake is also discussed including the rolling up of the vortices. In view of the complicated nature of the formulae involved, various simplifications are required before even the first approximation (beyond that provided by linearised theory) can be calculated for a concrete case. Accordingly it is not clear whether the results obtained in this way would represent

an improvement on linearised theory. The simple example which is given in the paper does not settle this point.

A. Robinson (Toronto, Ont.).

van Spiegel, E., and van de Vooren, A. I. On the theory of the oscillating wing in two-dimensional subsonic flow. Nationaal Luchtvaartlaboratorium, Amsterdam, Report **F. 142**, i+21 pp. (1953).

Timman's solution of the problem of the title [thesis, Delft, 1946; these Rev. **11**, 225], obtained by introducing elliptic cylinder coordinates and expanding the acceleration potential in Mathieu functions, is reproduced, and it is verified directly that the result satisfies the Possio integral equation. An error in Küssner's expression for the kernel of this integral equation [Luftfahrtforschung **17**, 370–378 (1940); these Rev. **2**, 330] is pointed out. *J. W. Miles.*

Lance, G. N. The kernel function of the integral equation relating the lift and downwash distributions of oscillating finite wings in subsonic flow. J. Aeronaut. Sci. **21**, 635–636 (1954).

The kernel function is expanded in powers and logarithms of the reduced frequency by taking the Laplace transform of its integral representation, expanding in inverse powers and logarithms of the spectrum parameter, and inverting term by term. The result agrees with that obtained more laboriously by Watkins, Runyan, and Woolston [NACA Tech. Note no. **3131** (1954); these Rev. **15**, 474]. The sonic case is treated separately. Finally, the method of steepest descent is used to obtain the leading term in the asymptotic development of the kernel for large reduced frequency.

J. W. Miles (Los Angeles, Calif.).

Hunn, B. A. A note on the evaluation of the supersonic downwash integral. Aeronaut. Quart. **5**, 111–118 (1954).

The downwash in the plane of symmetry of a delta wing having subsonic leading edges is expressed in terms of a single integral, the integrand of which contains incomplete, elliptic integrals. The method is essentially that of Lomax, Sluder, and Heaslet [NACA Rep. no. **957** (1950); these Rev. **12**, 453]. The author compares the limiting form of his result in the plane of the wing with that given by Ward [Aeronaut. Quart. **1**, 35–38 (1949)] and states that the latter is incorrect. *J. W. Miles* (Los Angeles, Calif.).

*****Sauer, R. Anwendung der Distributionstheorie auf das Problem des Überschall-Tragflügels.** Mémoires sur la mécanique des fluides offerts à M. Dimitri P. Riabouchinsky, pp. 289–308. Publ. Sci. Tech. Ministère de l'Air, Paris, 1954. 3000 francs.

L'auteur se propose de montrer comment la théorie des distributions de Laurent Schwartz peut être d'une grande utilité dans la résolution de certains problèmes de mathématiques appliquées. Les premières pages sont consacrées à une présentation simple des distributions de Dirac et des distributions "partie finie" tout à fait à la portée des ingénieurs. Pour illustrer l'utilisation de ces notions, l'auteur considère l'application de l'équation des ondes cylindriques à l'aérodynamique supersonique linéarisée. Le problème de Cauchy et le problème aux limites rencontré dans l'étude des ailes symétriques à portance nulle sont ainsi aisément résolus; naturellement les résultats sont identiques aux formules classiques. *P. Germain* (Paris).

Sedney, R. On Jones's criterion for thin wings of minimum drag. J. Aeronaut. Sci. **21**, 639–640 (1954).

Michelson, I. On the construction of a rational aerodynamic theory of slender bodies. U. S. Naval Ordnance Test Station, Inyokern, Calif., Tech. Memo. **1575**, v+24 pp. (1953).

Van Mieghem, J. La forme rotationnelle des équations de la dynamique atmosphérique et les invariants du mouvement de l'air. Arch. Meteorol. Geophys. Bioklimatol. Ser. A. **7**, 16–28 (1954).

The author starts by deriving the absolute vorticity equation for a system of curvilinear co-ordinates. This is then written in the form of a balance equation which expresses the rate of change of absolute vorticity in terms of its production and transport; in this way a definition of vorticity transport is deduced. It is shown that the transport of the kth component, Q^k, of the absolute vorticity is zero across the surface x^k=constant. This fact enables the author to reduce the vorticity equation to a balance equation in two dimensions. In addition it is shown that when there is no flow across the surface x^k=constant, Q^k is carried along the lines of flow. Two particular examples are then given: firstly spherical polar co-ordinates (λ, φ, r) and secondly a system of co-ordinates (λ, φ, ψ), where ψ is an arbitrary single-valued scalar quantity. In this second case the vorticity equation is combined with the equation of continuity to give Ertel's equation. Finally the balance equation is written in another form, which leads to a generalization of the Ertel-Rossby invariant. *M. H. Rogers.*

Erdélyi, Arthur. Variational principles in the mathematical theory of diffraction. Atti. Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **87**, 281–293 (1953).

This paper is concerned with the approximate solution of the problem of diffraction of monochromatic sound waves by a plane screen of finite dimensions. For definiteness, it is assumed that the wave function vanishes on the screen, though similar considerations can be applied to the other diffraction problems.

If the wave length is large compared with the dimensions of the obstacle, Rayleigh showed that the scattered field is very nearly a spherical wave whose amplitude is proportional to the electrostatic capacity of a conducting disc in the shape of the screen. But if the wave length is of the order of the dimensions of the screen, the variational method of Schwinger and Levine gives an approximation solution of the problem.

At first sight, the two methods appear quite distinct. Yet electrostatic capacity can be defined by a variational problem; and if this definition is applied to Rayleigh's method, the newer technique of Schwinger and Levine appears as the natural extension of the reformulated Rayleigh approximate solution. Moreover, although both methods are usually applied only to the determination of the distant field, it is shown here that the analysis will give the field at any point or any linear functional of the field. *E. T. Copson.*

Friedlander, F. G. Diffraction of pulses by a circular cylinder. Division of Electromagnetic Research, Institute of Mathematical Sciences, New York University, Research Rep. No. **EM-64**, ii+37 pp. (1954).

This paper deals with the diffraction of a sound pulse of small amplitude due to an instantaneous line source by a parallel rigid fixed circular cylinder. As it is impossible to give in a brief space an adequate account of this work, I merely quote the author's abstract.

"The diffracted field is resolved into terms representing diffracted pulses which have encircled the cylinder a number of times in either sense, and these in turn are expressed as a series of propagation modes with the same pulse fronts. Approximations valid near the front of a mode are then obtained. The corresponding results for an incident plane pulse are derived by a limiting process. It is found that a diffracted pulse has an essential singularity of a definite type at its fronts, similar to the initial stages of diffusion, and that the pressure rises very gradually at first. But after time intervals of the order of one-third to one-half of the radius of the cylinder divided by the velocity of sound, there seems to be no appreciable diffraction effect."

E. T. Copson (St. Andrews).

Copson, E. T. The reflexion of sound waves of finite amplitude by a rigid wall. Proc. Roy. Soc. London. Ser. A. 222, 254–261 (1954).

The motion considered is the one-dimensional motion of a polytropic gas of adiabatic exponent 5/3 filling the half-space on one side of a rigid plane wall with initial conditions of gas at rest and density proportional to $x^{3/2}$, where x is the distance from the wall. The isentropic equations of motion are integrated in closed form for general initial and boundary conditions and it is shown that the problem stated above does not admit a continuous isentropic solution. It is then assumed that the flow generates a non-uniform shock wave at the wall of such velocity that the flow behind the shock wave is still isentropic. A solution based on this assumption is found in closed form. The resulting shock wave moves out from the wall and has a velocity proportional to the time.

The author points out that the solution so obtained is not necessarily unique since there might be other such solutions with more than one shock wave. [The reviewer is of the opinion that the solution is unique. The particular combination of the initial density distribution and gas of this problem is of such a nature as to give rise to a limiting problem in the sense that it is possible to maintain the ratio ρ_1/ρ_0 constant, where ρ_0 is the density immediately in front of the shock and ρ_1 the density immediately in back. In general, the flow behind the non-uniform shock would be non-isentropic.]

P. Chiarulli (Providence, R. I.).

Elasticity, Plasticity

Jindra, F. Einige Anwendungen eines nichtlinearen Elastizitätsgesetzes. Ing.-Arch. 22, 121–144 (1954).

The author studies the theory proposed by Kauderer [Ing.-Arch. 17, 450–480 (1949); these Rev. 12, 63], a theory not essentially different from that of Voigt [Nachr. Ges. Wiss. Göttingen 1893, 534–552], whose incompleteness and indeed inconsistency was pointed out by Finger [Akad. Wiss. Wien, S.-B. IIa. 103, 163–200, 231–250 (1894)]. The author treats plane strain, the rotating cylinder, and the torsion of a body of revolution. The calculations, both algebraic and numerical, are formidable. [The author appears to be entirely unaware of modern developments in elasticity theory. For some of the problems he treats, fully general solutions are known; for others, numerical methods have been worked out for a general strain energy, or second-order effects have been calculated in a complete perturbation scheme. See chapter IV of the reviewer's paper [J. Rational Mech. Anal. 1, 125–300 (1952); 2, 593–616 (1953), these Rev. 13, 794; 15, 178], the work of Rivlin [ibid. 2, 53–81 (1953); these Rev. 14, 513], Rivlin and Topakoglu [see the following review], and the papers by Ericksen, Rivlin, Green, and others reviewed below.]

C. Truesdell (Bloomington, Ind.).

Rivlin, R. S., and Topakoglu, C. A theorem in the theory of finite elastic deformations. J. Rational Mech. Anal. 3, 581–589 (1954).

Assuming that the strain-energy function is a polynomial of degree $n+1$ in the displacement gradients, the authors show that the solution of any boundary-value problem (surface tractions given) can be reduced to the successive solution of n boundary-value problems in the classical linearized theory of elasticity. The procedure is valid for both isotropic and anisotropic materials. The method should be very useful for practical problems to investigate second and higher-order elastic effects.

W. Noll (Berlin).

Ericksen, J. L., and Rivlin, R. S. Large elastic deformations of homogeneous anisotropic materials. J. Rational Mech. Anal. 3, 281–301 (1954).

The authors consider the classical theory of finite elastic strain of anisotropic bodies. It is their merit to be the first to obtain the stress-strain relations appropriate to materials with internal constraints, and the first to determine the most general functional form of the strain energy for a particular anisotropic material. Their method rests on considering three constant mutually orthogonal unit vectors $H^\alpha{}_a$ indicating directions intrinsically defined in the undeformed material. Using the notations now standard in this subject [see the reviewer's paper in same J. 1, 125–171, 173–300 (1952); 2, 593–616 (1953); these Rev. 13, 794; 15, 178], they associate in the deformed material vectors $h^i{}_a$, defined by $h^i{}_a = H^\alpha{}_a x^i{}_{,\alpha}$. In terms of these they define

$$\Gamma_{ab} = C_{\alpha\beta} H^\alpha{}_a H^\beta{}_b = g_{ij} h^i{}_a h^j{}_b,$$

where $C_{\alpha\beta} = g_{ij} x^i{}_{,\alpha} x^j{}_{,\beta}$. For $c_{ij} = G_{\alpha\beta} X^\alpha{}_{,i} X^\beta{}_{,j}$, we get

$$(c^{-1})^{ij} = h^i{}_a h^j{}_a.$$

Also $C^{\alpha\beta} = \Gamma_{ab} H^\alpha{}_a H^\beta{}_b$. The authors consider constraints of the type $F_s(\Gamma_{ab}) = 0$, the F_s being assigned functions not more than five in number and such that $F_s(\delta_{ab}) = 0$. They remark that more general constraints have been considered in the past but are in general inconsistent with the symmetry of the stress tensor. For a body with strain energy $\Sigma = \Sigma(C_{\alpha\beta})$, by one of the above formulae we may write $\Sigma = \Sigma(\Gamma_{ab}, H^\alpha{}_a)$, and since the $H^\alpha{}_a$ are fixed in the material this gives

$$d\Sigma/dt = (\partial\Sigma/\partial\Gamma_{ab})(\partial\Gamma_{ab}/dt).$$

Putting this relation into the energy equation yields after some manipulation the definitive stress-strain relations

$$(*) \qquad t^{ij} = \left[2J^{-1} \frac{\partial\Sigma}{\partial\Gamma_{ab}} + \sum_s \lambda^s \frac{\partial F_s}{\partial\Gamma_{ab}} \right] h^i{}_a h^j{}_b,$$

λ^s being the Lagrangian multiplier corresponding to the constraint $F_s = 0$, and $J = |C^\alpha{}_\beta|^{\frac{1}{2}} = |\Gamma_{ab}|^{\frac{1}{2}}$. In the case of a homogeneous deformation $x^i = A^i{}_\alpha X^\alpha$ consistent with the constraints, (*) with the λ^s constant satisfies the equations of equilibrium.

Next the authors develop a formalism to express intrinsic symmetries (as for a crystal) in the theory of finite elastic

strain. Let the nine constants $\Lambda^a{}_b$ satisfy $\Lambda^a{}_c\Lambda^a{}_d=\delta_{cd}$, and put $\bar{H}^a{}_a=\Lambda^b{}_a H^a{}_b$, $\bar{\Gamma}_{ab}=C_{\alpha\beta}\bar{H}^\alpha{}_a\bar{H}^\beta{}_b=\Gamma_{cd}\Lambda^c{}_a\Lambda^d{}_b$. Then $\Sigma=\Sigma(\bar{\Gamma}_{ab})$. If this function $\Sigma(\bar{\Gamma}_{ab})$ is identical with that obtained by substituting $\bar{\Gamma}_{ab}$ for Γ_{ab} in $\Sigma(\Gamma_{ab})$, they say that Σ is "form invariant" with respect to the substitution $\Lambda^a{}_b$. They show that the set of substitutions with respect to which Σ is form invariant forms a group under multiplication defined by $\Lambda^a{}_b=\Lambda^a{}_{(1)c}\Lambda^c{}_{(2)b}$. The symmetries of a material are to be expressed by requiring that Σ be form invariant for specified $H^a{}_a$ and a specified group of substitutions. The authors apply this formalism only to the special case of complete rotational symmetry with respect to a specified direction $H^a{}_3$ and to the subcase, which they call "transverse isotropy", when the material exhibits symmetry with respect to reflection in any plane containing the vector $H^a{}_3$. The calculations are fairly elaborate. The result, when there are no constraints, is

$$t^i{}_j=2J^{-1}\left\{\left(\mathrm{II}\frac{\partial\Sigma}{\partial\mathrm{II}}+\mathrm{III}\frac{\partial\Sigma}{\partial\mathrm{III}}\right)\delta^i{}_j+\frac{\partial\Sigma}{\partial\mathrm{I}}(c^{-1})^i{}_j-\mathrm{III}\frac{\partial\Sigma}{\partial\mathrm{II}}c^i{}_j\right.$$
$$\left.+\frac{\partial\Sigma}{\partial\mathrm{I}'}h^i{}_3h_{3j}+\frac{\partial\Sigma}{\partial\mathrm{II}'}[(c^{-1})^i{}_kh_{j3}+(c^{-1})_{jk}h^i{}_3]h^k{}_3\right\},$$

where $\mathrm{I}'\equiv\Gamma_{33}$ and $\mathrm{II}'\equiv\Gamma_{3a}\Gamma_{a3}$, the other notations being standard.

Next the authors obtain a family of exact solutions for an incompressible transversely isotropic body. Let R, Θ, Z be cylindrical co-ordinates so chosen that $H^R{}_3=H^\Theta{}_3=0$, $H^Z{}_3=1$. The assumed deformation is

$$r=(AR^2+B)^{\frac{1}{2}},\quad \theta=C\Theta+DZ,\quad z=E\Theta+FZ,$$

where r, θ, z are again cylindrical co-ordinates and where $A(CF-DE)=1$, $AR^2+B>0$, for some R. The authors show that this deformation can be produced by surface tractions only, and they calculate the appropriate system of stresses. Even for the isotropic case, this solution includes and generalizes several known results. The authors discuss it only for three of the special cases included: simultaneous extension and torsion of a solid cylinder, simultaneous inflation and elongation of a hollow cylinder, dislocation of cylinder by removal of a wedge. Next the authors obtain solutions for the bending of a block in two cases: the axis of isotropy in the undeformed material is normal to the faces which after deformation become cylindrical, or it lies in these faces. In the latter case the results are simpler when the axis is perpendicular to one of the faces; otherwise, shearing forces on the four plane faces are required in order to effect the flexure. Finally the authors generalize the known solution for a rotating cylinder, possibly hollow, to the case of a material with transverse isotropy about the axis of revolution. *C. Truesdell* (Bloomington, Ind.).

Green, A. E. A note on second-order effects in the torsion of incompressible cylinders. Proc. Cambridge Philos. Soc. 50, 488–490 (1954).

Green and Shield [Philos. Trans. Roy. Soc. London. Ser. A. 244, 47–86 (1951); these Rev. 13, 509] have constructed a general theory of second-order effects in the torsion of an incompressible elastic cylinder of uniform cross-section, reducing it to the solution of certain integral equations. Rivlin [J. Rational Mech. Anal. 2, 53–81 (1953); these Rev. 14, 513] has shown how second-order effects in any elastic body may be calculated from the solutions of two appropriate problems in the linear theory. For the torsion problem, the torsion function in the linear theory must thus be known if we are to calculate all the second-order effects. By use of Betti's theorem, however, Rivlin was able to calculate the fractional elongation of a compressible cylinder explicitly in terms of the moment of inertia, the classical torsional rigidity, and the five elastic constants of first and second order. He then obtained the corresponding result for an incompressible cylinder by a limiting process. The author shows that the method of Green and Shield can also yield this explicit result without our having to solve the integral equations. He then generalizes it to the case when the cylinder is stretched before it is twisted. *C. Truesdell.*

Green, A. E., and Spratt, E. B. Second-order effects in the deformation of elastic bodies. Proc. Roy. Soc. London. Ser. A. 224, 347–361 (1954).

Rivlin [J. Rational Mech. Anal. 2, 53–81 (1953); these Rev. 14, 513] has embedded the solution of a boundary value problem in the linear theory of elasticity as the first approximation in a perturbation procedure for finding a certain (possibly not unique) solution of a corresponding problem in the general theory of finite strain. His method rests upon expansions in powers of the displacement gradients. [In the paper cited, only the second approximation and only the isotropic case are considered, but in the work by Rivlin and Topakoglu reviewed third above the method is formulated in the nth approximation for general elastic materials.] The authors propose an alternative scheme based upon expansions of all quantities in power series in a parameter ϵ. They work out the details for the second order approximation for isotropic bodies. Their method has an advantage over Rivlin's in that it applies equally to incompressible or compressible bodies, while Rivlin's method applies only to compressible bodies and yields results in the incompressible case only by a limiting process. However, the authors obscure this advantage by describing their procedure in detail separately for the two cases. Their results do not seem to yield Rivlin's elegant reduction of the entire problem to the solution of a finite number of appropriate problems in the linear theory, and the calculations seem formidable. The authors illustrate their method by calculating second order effects in the torsion of an incompressible solid of revolution, specializing to the case of a cone. *C. Truesdell* (Bloomington, Ind.).

Baker, M., and Ericksen, J. L. Inequalities restricting the form of the stress-deformation relations for isotropic elastic solids and Reiner-Rivlin fluids. J. Washington Acad. Sci. 44, 33–35 (1954).

The reviewer [J. Rational Mech. Anal. 1, 125–171, 173–300 (1952); 2, 593–616 (1953); these Rev. 13, 794; 15, 178] has raised the question of what restrictions on the form of the strain energy Σ of an elastic solid or the form of the generalized viscosities in a Reiner-Rivlin fluid are imposed by such irreversibility principles as may be valid for continuous media in general. This is in fact the major open problem in the principles of elasticity and fluid mechanics today. The reviewer, by considering a special class of deformations, has given an argument indicating that in an incompressible isotropic elastic solid we should always have

$$(*)\qquad (1+\delta_i)^2\frac{\partial\Sigma}{\partial\mathrm{II}}+\frac{\partial\Sigma}{\partial\mathrm{I}}\geqq 0,$$

where δ_i is any principal extension. Ericksen [ibid. 2, 329–337 (1953); these Rev. 14, 1147] has shown that (*) is necessary and sufficient that second-order waves propagate

at real speeds. The authors make a major contribution to this problem by relating it to a purely algebraic theorem.

Let the real symmetric square matrix **A** be an analytic isotropic function of the real symmetric square matrix **B**. By the theorem of Reiner [Amer. J. Math. 67, 350–362 (1945); these Rev. 7, 44],

$$(^{**})\qquad \mathbf{A}=f_0\mathbf{1}+f_1\mathbf{B}+f_2\mathbf{B}^2,$$

where f_0, f_1, f_2 are functions of the principal invariants of **B**. Then to a proper number b_i of **B** the formula

$$a_i=f_0+f_1b_i+f_2b_i^2$$

sets in correspondence a unique proper number a_i of **A**. The authors show that a necessary and sufficient condition that $b_i>b_j$ imply $a_i>a_j$ is

$$(^{***})\qquad f_1+f_2(b_i+b_j)>0.$$

Taking up elasticity, the authors state that it is reasonable to suppose that in an isotropic body the greatest (least) tension occurs in the direction corresponding to the greatest (least) extension. An analytical expression for this postulate is $t_i>t_j$ if and only if $\delta_i>\delta_j$. By applying their algebraic theorem, the authors show that their postulate is equivalent to (*), except that $\geqq$ is to be replaced by $>$ if the three principal extensions are all unequal. [Experiments on rubber yield the stronger inequalities $\partial\Sigma/\partial\mathrm{I}>0$, $\partial\Sigma/\partial\mathrm{II}>0$; these the reviewer strove unsuccessfully to derive from theory.]

Next the authors consider Reiner-Rivlin fluids [see the reviewer, op. cit., ch. V]. They assert that it is reasonable to suppose that the greatest (least) tension will be exerted across a plane whose normal is in the direction in which the rate of extension is greatest (least). An analytical expression of this postulate is $t_i>t_j$ if and only if $d_i>d_j$. Since the appropriate constitutive equation is (**) with $\mathbf{A}=\mathbf{T}$, $\mathbf{B}=\mathbf{D}$, the postulate is equivalent to (***) with the corresponding change of notation. [Here the only available results of experiment seem to indicate only $f_1>0$, $f_2>0$, inequalities apparently unrelated to the authors'.] For a non-rigid plane motion of an incompressible fluid, (***) reduces to $|f_2d_1/f_1|<1$. From a theorem of Ericksen [Z. Angew. Math. Physik 4, 260–267 (1953); these Rev. 15, 172] it follows that in such a fluid the characteristic directions are never real.

The authors' methods and results, both in generality and in logical clearness, constitute an advance over the reviewer's. However, like his, they are purely mechanical. The final inequalities must eventually come from thermodynamics, not yet formulated in sufficient generality nor successfully applied to modern continuum mechanics.

C. Truesdell (Bloomington, Ind.).

Babič, V. M. On the equations of motion of a nonlinearly elastic medium. Doklady Akad. Nauk SSSR (N.S.) 97, 41–44 (1954). (Russian)

The author determines the velocities of propagation of acceleration waves in an elastic material for which the strain energy is the sum of two functions, one being an essentially arbitrary function of the intensity of the deviator of the infinitesimal strain tensor, the other being the strain energy associated with change of volume in the linear theory of elasticity. This includes certain theories of plasticity. He discusses reality of the velocities and points out that waves corresponding to one of the velocities are transverse. *J. L. Ericksen* (Washington, D. C.).

Courtaigne, O. Le paradoxe du talus élastique. Ann. Ponts Chaussées 124, 465–473 (1954).

The author considers a semi-infinite solid bounded by two planes, $\beta+\pi/2$ being the angle formed by the planes. The solid is assumed to be in a state of plane stress, the plane of the stress being perpendicular to the bounding planes, and a constant body force acts in this plane, perpendicular to one bounding plane. He is concerned with solutions of the equations of linear elasticity leaving the bounding planes free of surface tractions. To determine a unique solution, one must specify suitable conditions at infinity, but the author seems to feel that St. Venant's principle should imply that the solution in the vicinity of the intersection of the bounding planes should not be greatly affected by changes in the conditions at infinity. He obtains a solution satisfying the boundary conditions on the bounding planes and concludes that, for some values of β, the solution does not yield results in accord with common sense. Assuming there are other solutions which do, one is led to conclude that an improper application of St. Venant's principle has been made. The author regards this as paradoxical. Since Steinberg's analysis of St. Venant's principle [Quart. Appl. Math. 11, 393–402 (1954); these Rev. 15, 370] does not imply the validity of the author's application, and since the reviewer knows of no other mathematical results which do, there seems to be no sound reason for regarding the author's results as paradoxical. *J. L. Ericksen.*

Varvak, P. M. A method of approximate solution of spatial problems of the theory of elasticity. Dopovidi Akad. Nauk Ukrain. RSR 1953, 285–288 (1953). (Ukrainian. Russian summary)

In the solution of three-dimensional problems of elasticity by the use of the Galerkin method, it is suggested that the biharmonic equation for the Galerkin function and the representations of the components of stress in terms of the derivatives of the Galerkin function be replaced by difference equations on a discrete point-lattice. Boundary conditions are discussed and it is noted that the role of the stress function in these equations is independent of the loading and hence the derived figures of influence for a given body can be used for all loadings. *C. Saltzer.*

Hardiman, N. Jessie. Elliptic elastic inclusion in an infinite elastic plate. Quart. J. Mech. Appl. Math. 7, 226–230 (1954).

The author considers the two regions of the plane defined by the ellipse $x^2/a^2+y^2/b^2=1$ and assumes that the two regions are made of different elastic materials. Using the potentials $\Omega(z)=Az$, $\omega(z)=Bz^2$ $(z=x+iy)$ for the inside of the ellipse and $\Omega(z)=Cz+cF\zeta^{-1}$, $\omega(z)=Gz^2+Hc^2\log\zeta+Jc^2\zeta^{-2}$, with $z(\zeta)=c(\zeta+\lambda\zeta^{-1})$, $0<\lambda<1$, $a=c(1+\lambda)$, $b=c(1-\lambda)$, for its outside, he shows that for any given set of values of any system of constants A, B and G, H, J the values of the other can be determined in such a way that the displacement and stresses $\widehat{nn}$, $\widehat{ns}$ across the elliptic boundary are continuous. Besides this, an expression for the discontinuity of the corresponding peripheral stresses across the boundary is given.

C. Arf (Istanbul).

Nocilla, Silvio. Sul problema della piastra a contorno epicicloidale incastrata. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. 87, 79–89 (1953).

The author writes the solution of $\nabla^4w-\lambda^4w=0$ in the form

$$\sum[c_kJ_k(\lambda r)+d_kI_k(\lambda r)]\begin{matrix}\cos\\ \sin\end{matrix}\,k\theta,$$

and subjects this solution to the homogeneous boundary conditions $w=\partial w/\partial n=0$ on the boundary of the epicycloid $x+iy=re^{i\theta}=nae^{i\phi}+be^{in\phi}$. The resulting system of homogeneous equations leads to an infinite determinantal equation for λ, and the author proves that the infinite determinant is convergent. *A. Erdélyi* (Pasadena, Calif.).

Müller, Wilhelm. Über die Darstellung der Durchbiegung von rechteckigen Platten durch Integrale der ϑ-Logarithmen. Z. Angew. Math. Mech. **34**, 12–18 (1954). (English, French and Russian summaries)

Closed expressions are found for the deflection of (i) a rectangular plate supported at the corners and under uniform pressure, (ii) the plate under a concentrated force. The series which give the deflection turn out to be summable in terms of integrals of logarithmic theta functions.

L. M. Milne-Thomson (Greenwich).

Müller, W. Beitrag zur Biegungstheorie der Mehrpilzplatte. Österreich. Ing.-Arch. **8**, 1–10 (1954).

A rectangular plate (flat slab floor) under uniform transverse loading is supported by k^2 thin columns of rectangular cross-section symmetrically arranged with respect to the plate center-lines. Edge conditions prescribe zero slope and zero shear. The resultant load on the plate (uniform load minus column reactions) is represented by a Fourier series, and the Fourier expansion for the deflections is found by the method of undetermined coefficients. Series for bending moments and particularly the moment sum are found by differentiation. It was found necessary to treat separately the cases where k^2 is an odd or an even number. No numerical work is included, nor, unfortunately, are any specific references to other work on this subject given; although the author does refer to his own previous papers and results, he does not mention where they can be found. [Other treatments of this problem are described in S. Timoshenko, Theory of plates and shells, McGraw-Hill, New York, 1940, Section 46.] *W. Nachbar* (Seattle, Wash.).

Yeh, Kai-yuan. Large deflection of a circular plate with a circular hole at the center. Acta Sci. Sinica **2**, 127–144 (1953).

A treatment of the problem of transverse bending of a circular ring plate with built-in outer edge and with inner edge displaced transversely, without change of slope and horizontal displacement. The basic equations are the same as those used by Way [Trans. A.S.M.E. **56**, 627–636 (1934)] and others. The paper uses two terms of a perturbation-series solution (the first term representing the result of linear theory) and it appears as if this two-term approximation has been used for the purpose of computation without consideration of its practical range of usefulness. The author's criticism of Way's power-series method appears to this reviewer as unjustified. It should further be remarked that the same problem has previously been solved by power-series methods [K. Federhofer, Österreich. Ing.-Arch. **1**, 21–35 (1946); these Rev. **8**, 241], that the corresponding membrane solution, ascribed to a Russian author in 1951, is due to Schwerin [Z. Tech. Phys. **10**, 651–659 (1929)] and that a perturbation solution as well as an asymptotic solution of a very similar problem has previously been discussed by the reviewer [Quart. Appl. Math. **10**, 167–173 (1952); these Rev. **14**, 429]. *E. Reissner* (Cambridge, Mass.).

Wasel, A. D. A method of determining plate bending by use of a punched-card machine. J. Assoc. Computing Mach. **1**, 105–110 (1954).

Kaplan, A., and Fung, Y. C. A nonlinear theory of bending and buckling of thin elastic shallow spherical shells. NACA Tech. Note no. **3212**, 58 pp. (1954).

This is the report of an extensive program of theory and experiment on the buckling of shallow spherical domes. The authors begin by criticising the equations of E. Reissner [J. Math. Physics **25**, 80–85 (1946); 279–300 (1947); these Rev. **7**, 502; **8**, 359] because they are linearized. To replace them, they follow a method which for the case of a flat plate was used by W. Z. Chien [Chinese J. Phys. **7**, 102–113 (1947); these Rev. **9**, 481]; thus they obtain a formidable non-linear system. For this system they calculate the classical buckling load when the edge is clamped and the lateral load is symmetric. Their method, like Chien's, is to expand all quantities in powers of the center deflection ratio. The first resulting equation they solve explicitly in terms of ber, bei, ker, and kei functions, but for the higher approximations they have to use numerical methods.

As is well known, the classical buckling load calculated from linearized equations is too large, and to replace it Kármán and Tsien proposed an energy criterion. The details of Tsien's subsequent modification of this theory and of the calculation within it are open to objection. For example, in applying it to the present case the authors find that it yields critical loads which are higher rather than lower than those from the classical linearized theory which it is supposed to replace. In principle, the energy criterion can never yield a higher buckling load than the classical one, and the foregoing paradox results only from mathematical errors ("approximations"). It was the authors' desire to calculate the two buckling criteria correctly and to compare them on their merits. This they were unable to do. First, while they could see inaccuracies in Tsien's formulation of his idea, they found that to improve it would be a major endeavor. Second, in their treatment of the classical theory without linearizing it they encountered difficulties of convergence. Limiting attention to sufficiently small values of a parameter of shallowness, they find that the classical theory yields buckling loads smaller than those obtained from the linearized theory and usually called "classical", in fact as much as 50% smaller in the range where their method works. For less shallow shells they are able to calculate curves of deflection at loads below the critical load; for some cases the maximum deflection occurs approximately halfway between the center and the edge of the shell.

From their measurements they conclude that the classical (non-linear) theory is in agreement with experiment for their range of calculation. They suggest that the well-known discrepancies are due entirely to an improper mathematical process (linearization), not to any defect in the theory itself. However, they say that there appears to be a trend toward the energy criterion for somewhat less shallow shells.

C. Truesdell (Bloomington, Ind.).

Naghdi, P. M., and Berry, J. G. On the equations of motion of cylindrical shells. J. Appl. Mech. **21**, 160–166 (1954).

The authors set up equations for the bending of cylindrical shells. They follow one of the usual methods: they neglect the ratio (normal distance/radius) and they omit certain terms in the stress-strain relations, in effect taking the stress as linear in the normal distance. Thus they obtain a system of three partial differential equations for the components of displacement. By successive differentiations they are able to obtain a single equation of eighth order for the normal

displacement alone and equations of fourth order for the other two displacements when the solution to the first is known. These three equations occupy half a page. To compare the results with those of other theories, they calculate the characteristic equation corresponding to separated variables. They find their equation differs from those of Flügge and Byrne, Vlasov, and Kennard. They conclude: "we have not succeeded in arriving at definite conclusions concerning the relative merits of the different approximations. . . . However, . . . there appears to be little advantage in going beyond Love's first approximation." [The reviewer is unable to draw this conclusion, or any definite conclusion, from the analysis presented. For the case of a shell of revolution, he has pointed out an inconsistency in a derivation of the type given here [Trans. Amer. Math. Soc. 58, 96–166 (1945), see p. 111; these Rev. 7, 231]. Moreover, the authors themselves note that their eq. [3]$_3$ is an identity following from the definitions of the resultants, but that their "approximate" formulae [11] do not satisfy it. This objection does not apply to the theories of Flügge, Byrne, and the reviewer, which rest upon making specified and definite assumptions at the outset, once and for all, and then proceeding to an exact treatment of the resulting equations.] *C. Truesdell.*

Rüdiger, D. Ein Beitrag zum Randstörungsproblem isotroper Kreiszylinderschalen. Ing.-Arch. 22, 160–162 (1954).

Zerna [Ing.-Arch. 20, 357–362 (1952); these Rev. 14, 700] derived a simplified theory of the bending of circular cylindrical shells, based upon a pair of differential equations. The author reduces these to a single equation of eighth order, for which he determines the form of solution by separation of variables. *C. Truesdell.*

Layrangues. Théorie de la statique des voiles minces par les méthodes de l'analyse vectorielle. Application aux voiles en paraboloïde hyperbolique, en hyperboloïde de révolution à une nappe et en cône de révolution. Ann. Ponts Chaussées 123, 69–105 (1953).

Layrangues. Théorie de la statique des voiles minces en coordonnées rectilignes. Ann. Ponts Chaussées 124, 27–67 (1954).

These papers derive and illustrate in special cases Love's classical theory of shells in the extensional or "membrane" approximation. The author scrupulously avoids reference to any work not published in the same journal and thus is able to discover well known facts with gusto. Many problems are solved and discussed in detail. Apparently the author restricts himself to statically determinate problems where special symmetries permit an explicit solution in terms of elementary functions. The collection so obtained is the largest known to the reviewer; apart from awkward notations, it should help those studying shapes and loads of practical usefulness. *C. Truesdell* (Bloomington, Ind.).

Mikeladze, M. Š. The bending of a beam stretched by centrifugal forces. Akad. Nauk SSSR. Inženernyĭ Sbornik 16, 173–176 (1953). (Russian)

Hu, Hai-chang. Torsion of prisms bounded by two intersecting circular cylinders. Acta Sci. Sinica 2, 269–281 (1953).

Using bipolar coordinates and Fourier's integral the problem of the torsion of prisms bounded by two intersecting circular arcs is discussed. It is shown that when the angle of intersection of the two arcs is commensurable with π the stress function and consequently the maximum shearing stress, and also the torsional rigidity are expressible in closed form in terms of elementary functions. The case when the two circular arcs are orthogonal is treated as a special case. *R. M. Morris* (Cardiff).

Abramyan, B. L. On the problem of torsion of nonhomogeneous prismatic bars. Akad. Nauk Armyan. SSR. Doklady 14, 9–14 (1951). (Russian. Armenian summary)

An approximate solution, based on the theorem of minimum potential energy, is outlined for the Saint-Venant torsion problem for beams with multiply connected cross-sections. It is supposed that the elastic medium is reinforced along the contours of the cross-section by thin layers of material with high shearing modulus. An example is worked out for the special case of the hollow circular pipe, but no attempt is made to estimate the accuracy of approximations. *I. S. Sokolnikoff* (Los Angeles, Calif.).

Yeh, Kai-yuan. St. Venant's torsion problem with stress function in the form of a third degree polynomial. Acta Sci. Sinica 2, 282–300 (1953).

The conjugate function ψ of the torsional function φ is expressed as a third-degree polynomial. The closed boundary formed from the boundary condition of ψ is shown to take the following forms: (i) an equilateral triangle, (ii) the lines of the shearing stress of the previous equilateral triangle, (iii) a straight line and one of the branches of a hyperbola, and the lines of shearing stress in this case, (iv) a curve of the third degree. The shearing stress components, the maximum shearing stress, and the torsion moment are found in these cases. *R. M. Morris* (Cardiff).

Wittrick, W. H. Buckling of oblique plates with clamped edges under uniform shear. Aeronaut. Quart. 5, 39–51 (1954).

In a previous paper [same Quart. 4, 151–163 (1953); these Rev. 14, 927] the author derived by the energy method of Rayleigh a determinantal equation for obtaining the critical magnitude of any uniform system of edge stress applied to a clamped oblique plate. In the present paper this equation is used to derive values for the buckling stresses of clamped oblique plates subjected to pure shear along and perpendicular to two edges of the plate. As would be expected, it is found that reversal of the direction of the shear produces a change in the critical value. The lower value occurs when the shear is tending to increase the obliquity of the plate and critical values corresponding to this direction are calculated for 45° oblique plates with side ratios varying between 0.60 and 1.67. The critical values obtained for the reverse direction were obviously very inaccurate, due to the inadequacy of the series which was used to represent the buckling mode. As yet no satisfactory method has been found for overcoming this difficulty. It is shown for 45° oblique plates with clamped edges that the maximum ratio between the larger and smaller critical shear stresses is 2.69, as compared with 8.00 when edges are simply supported. These are the limiting values for very short plates. *R. Gran Olsson* (Trondheim).

Buckens, F. Théorie limite du flambage d'une plaque circulaire chauffée en son centre. Ann. Soc. Sci. Bruxelles. Sér. I. 68, 63–71 (1954).

Let h be the thickness of a circular plane plate of radius R. The equation of the symmetrical deflection w of the plate is

$$\text{(1)}\qquad \varphi''+\varphi'/r-\varphi/r^2=N_r/D-\varphi \quad (\varphi=dw/dr),$$

N_r being the radial tension integrated over the thickness, $D=Eh^3/12(1-\nu^2)$, E the modulus of elasticity and ν Poisson's ratio. If the plate is submitted to a temperature T, which depends only on the distance r from the centre of the plate and which is supposed to be uniform in each point of the thickness, one has [S. Timoshenko and J. N. Goodier, Theory of elasticity, McGraw-Hill, New York, 1951, pp. 399–438; these Rev. 13, 599]

$$\text{(2)}\qquad N_r=\alpha Eh\left(\int_0^R Trdr/R^2-\int_0^R Trdr/r^2\right),$$

α designating the coefficient of thermal expansion. The solution of the equation obtained by substitution of (2) into (1) for different temperature distributions and different boundary conditions was discussed by the author [Eighth Congress of Theoretical and Applied Mechanics, Istanbul, 1952]. In this paper the equation is treated only for the very special assumption of constant temperature (for instance zero), with exception of the neighbourhood of the centre where the temperature may obtain great values. For $r_0\ll R$, one has (3) $\int_0^{r_0}Trdr=Tr_0^2/2$ and for $r_0<r\leqq T$, $T=0$, yielding the equation

$$\text{(1a)}\qquad \varphi''+\varphi'/r-\left(\frac{\lambda}{R^2}+\frac{1-\lambda}{r^2}\right)\varphi=0,$$

the parameter $\lambda=6\alpha(1-\nu^2)Tr_0^2/h^2$ being positive. The solution is given by Bessel functions of the first and second kind with the parameter $n=(1-\lambda)^{1/2}$, argument $r\lambda^{1/2}/R$, and is extensively investigated for clamped edge ($\varphi(R)=0$) and for simple support ($\varphi'(R)+\nu\varphi(R)=0$) along the periphery. At the end of the paper some numerical examples are involved. (There are some misprints: In the formula of the paramter λ on p. 64 the coefficient α is dropped, deflection W instead of w on p. 63.) *R. Gran Olsson* (Trondheim).

Goodier, J. N., and Hsu, C. S. Nonsinusoidal buckling modes of sandwich plates. J. Aeronaut. Sci. 21, 525–532 (1954).

In previous papers wrinkling and "quasi-Euler" buckling modes of sandwich plates have been taken as sinusoidal. In this report the possibility of nonsinusoidal modes is investigated. The core slab material considered is the special orthotropic one which does not resist deformation in its own plane, but only extension and shearing perpendicular to its plane. This choice is made for two reasons: (1) such an idealized core material has been adopted by previous investigators as closer to certain real cores than the isotropic material, and also because the "in the plane" stress components are believed to play a minor part in the core action; (2) it permits a relatively simple analysis of nonsinusoidal deformations and the satisfaction of end conditions which would present formidable analytical difficulties if the core were isotropic. It is found that the critical wrinkling loads corresponding to sinusoidal modes are not the lowest consistent with simply supported ends. Loads about half as great correspond to buckling modes in which the deformation is confined to the end zones. The main body of the panel remains flat but undergoes a lateral displacement. The effect of clamping the ends is also examined. The pattern of the theory follows closely, though not exactly, that of the long column with elastic foundation considering end conditions other than simple support. K. Ratzersdorfer [Die Knickfestigkeit von Stäben und Stabwerken, Springer, Wien, 1936, p. 141] showed that, when the ends are allowed to deflect, the critical load is halved and the deflection localized at the ends. The problem is the same when the ends are pinned but the rigid base of the elastic foundation is freed to "float" under no forces other than those of the springs representing the foundation. This is a close analog of the sandwich problem. *R. Gran Olsson* (Trondheim).

Federhofer, K. Knicklast der axial gedrückten Kreiszylinderschale bei Vorhandensein eines entlang des Zylindermantels veränderlichen elastischen Widerstandes. Österreich. Ing.-Arch. 8, 90–97 (1954).

The buckling load of an axially compressed thin cylindrical shell can be estimated by means of the results of the well known stability investigations of such shells. They are valid for constant wall thickness, but can be extended also to the case of continuously changing wall thickness [K. Federhofer, same Arch. 6, 277–288 (1952); these Rev. 14, 928]. The buckling load is increased by an elastic bedding completely surrounding the shell of the cylinder. In the present paper this increase is calculated for a constant bedding coefficient, and also for a coefficient increasing linearly along the generatrix of the cylinder whereby the elastic bedding is supposed to react according to the classical hypothesis of Heinrich Hertz (proportionality of deflection and reaction). *R. Gran Olsson* (Trondheim).

Jung, E. Ein Beitrag zur Dynamik der Drahtseile. Z. Angew. Math. Mech. 34, 66–68 (1954).

The paper is concerned with the nonlinear vibrations of a load carried by a steel cable that is wound on a drum rotating with constant angular velocity. The author assumes that the length and the diameter of the cable are quadratic functions of the true stress in the cable. The equation of motion is therefore nonlinear and its approximate solution is found by using an iteration process. A numerical example is given at the end of the paper. *E. T. Onat* (Ankara).

Satô, Yasuo. Study on surface waves. XI. Definition and classification of surface waves. Bull. Earthquake Res. Inst. Tokyo 32, 161–168 (1954). (Japanese summary)

Stoneley, R. Rayleigh waves in a medium with two surface layers. I. Monthly Not. Roy. Astr. Soc. Geophys. Suppl. 6, 610–615 (1954).

The period equation is obtained in the form of a determinant of the tenth order. If the length of Rayleigh waves propagating in this layered half-space is very small, the determinant reduces to the product of three determinants. They yield then period equations for very short Rayleigh waves along the free surface as well as for similar waves along the two interfaces. *W. S. Jardetzky.*

Thomas, T. Y. Interdependence of the yield condition and the stress-strain relations for plastic flow. Proc. Nat. Acad. Sci. U. S. A. 40, 593–597 (1954).

The author investigates the structure of the stress-strain relations of a perfectly plastic solid under the following assumptions: 1) the solid is incompressible, 2) its mechanical behavior is characterized by a relation between the stress deviation s_{ij} and the velocity strain ϵ_{ij} that is invariant under proper orthogonal coordinate transformations, and 3) this relation does not establish a one-to-one correspondence between the components of these tensors. From 1) and 2) it follows that

$$s_{ij}=G\epsilon_{ij}+H(\epsilon_{ik}\epsilon_{jk}-\tfrac{1}{3}\epsilon_{kl}\epsilon_{kl}\delta_{ij}),$$

where G and H are functions of the scalar invariants

$$\xi=\epsilon_{ij}\epsilon_{ij}, \quad \zeta=\epsilon_{ij}\epsilon_{jk}\epsilon_{ki}.$$

This result, which the author attributes erroneously to Rivlin and Erickson, is due to Reiner [Amer. J. Math. 67, 350–362 (1945); these Rev. 7, 44] and this reviewer [J. Appl. Phys. 15, 65–71 (1944)]. If, in particular, it is assumed that H vanishes while G depends only on ξ, the relation $s_{ij}=G\epsilon_{ij}$ yields $s_{ij}s_{ij}=G^2\xi$. It follows then from 3) that $G^2\xi$ must be constant and the stresses must satisfy the yield condition of von Mises. The general case is treated along similar lines, and the argument is also presented in terms of the principal components of stress deviation and velocity strain. A serious restriction of the validity of the author's results stems from the fact that all invariants are assumed to be of class C^1 with respect to their arguments. Thus, the author finds that Tresca's yield condition is coupled with a flow law that stipulates proportionality between stress deviation and velocity strain, whereas this yield condition is more naturally associated with a flow law that predicts simple shear whenever the three principal shearing stresses are distinct [see, for instance, Koiter, Quart. Appl. Math. 11, 350–354 (1953); these Rev. 15, 583]. *W. Prager.*

Green, A. P. Hodographs in problems of plane plastic stress. J. Mech. Phys. Solids 2, 296 (1954).

In this note, the author points out that a relation between the velocity characteristics in the problem of plane plastic strain and the hodograph map of these characteristics remains valid for problems of plane plastic stress. The relation is the following: a velocity characteristic direction at any point is orthogonal to the hodograph of this direction. This result follows from the fact that velocity characteristics are along zero extension rate directions. *N. Coburn.*

Green, A. P. On the use of hodographs in problems of plane plastic strain. J. Mech. Phys. Solids 2, 73–80 (1954).

The author considers the hodograph or the velocity field of an ideal plastic body undergoing plane plastic deformation. First, some known basic relations satisfied by the velocity vector are summarized. These include an expression for the maximum shear strain-rate in terms of the curvatures of the slip lines and the corresponding hodograph lines. All of these relations are of value in the graphical determination of the hodograph. Finally, the author considers some problems in which the slip-line field and the hodograph are identical. These include several problems of compression between rough parallel dies. The slip lines and the corresponding hodograph lines are drawn and in one problem some details of the analysis are furnished. *N. Coburn.*

Hill, R. On the limits set by plastic yielding to the intensity of singularities of stress. J. Mech. Phys. Solids 2, 278–285 (1954).

The author considers a material whose mechanical properties are not specified beyond the statement that appreciable deformation of an element is possible only if the local stresses satisfy a certain relation (such as the yield condition of a perfectly plastic solid). The paper deals with the question: what distributions of surface tractions over some given part S_1 of the surface of a body made of this material produce significant deformation regardless of the tractions and constraints on the remainder S_2 of the surface? If the given stress condition is a convex function of the stress components, a critical distribution of surface tractions over S_1 is obtained by treating this stress condition as the yield condition and plastic potential of a perfectly plastic solid which has the shape of the given body and is rigidly constrained over S_2. Any distribution of surface tractions over S_1 that produces yielding in this plastic body is a critical distribution for the given body. The methods of limit analysis can thus be applied to the problem which, at first glance, would seem to be outside the scope of the mathematical theory of plasticity. Particular attention is given to critical distributions of surface tractions near the vertex of a wedge. This problem is treated directly as well as by the method outlined above. *W. Prager.*

Lomakin, V. A. On large elastic-plastic deformations. Vestnik Moskov. Univ. Ser. Fiz.-Mat. Estest. Nauk 9, no. 5, 41–45 (1954). (Russian)

The paper presents an analysis for the finite deformation of a thick cylinder subject to combinations of internal and external pressure. Plane-strain conditions are assumed and the material is supposed to be incompressible. A total strain type plastic law is used together with the natural (logarithmic) strain measure. The longitudinal thrust is found for the general case. For the case of internal pressure only, an upper bound is found to the rate of increase of pressure with expansion. This bound is said to have led to accurate prediction of the level of deformation at which instability was observed in an aluminium tube ($b/a=2$).
R. M. Haythornthwaite (Providence, R. I.).

Moskvitin, V. V. On elastic-plastic bending of a beam. Vestnik Moskov. Univ. Ser. Fiz.-Mat. Estest. Nauk 9, no. 5, 33–40 (1954). (Russian)

The simple plastic theory for bending of beams is applied to establish moment-curvature relations for rectangular beams composed of ideal elastic-linearly strain hardening material and subject to monotonic or cyclic loading programs.
R. M. Haythornthwaite (Providence, R. I.).

Kostyuk, A. G. Stresses in a continuous rotating cylinder beyond the elastic limit. Akad. Nauk SSSR. Prikl. Mat. Meh. 18, 453–456 (1954). (Russian)

The author examines the effect of elastic compressibility on the limiting speed of rotating cylinders. Expressing stress as a series expansion of powers of a small parameter, the first two sets of stress correction terms are evaluated for a general total strain type law. Specializing to a case in which effective stress is proportional to the nth power of the effective-strain, it is shown that compressibility reduces the limiting speed, except when n is large. No numerical values are given. *R. M. Haythornthwaite.*

Torre, C. Kritik und Ergänzung des Maxwellschen Ansatzes für elastisch-zähe Stoffe. Verdrehung von Stäben als Beispiel. Österreich. Ing.-Arch. 8, 55–76 (1954).

The author gives some motivation for using Natanson's equations of visco-elasticity. [He does not appear to be familiar with Natanson's work or with the severe and just criticism of it by Zaremba; the controversy is contained in several papers in the Bull. Internat. Acad. Sci. Cracovie. Cl. Sci. Math. Nat. for 1901–1904.] He observes that time differentiation of the stress-strain relations of elasticity theory introduces a term depending on the local rotation, but he neglects a number of other terms. [For the exact analysis, see §55 of the reviewer's paper in J. Rational Mech. Anal. 1, 125–171, 173–300 (1952); and the revised version, ibid. 2, 593–616 (1953); these Rev. 13, 794; 15, 178.] He applies his equations to the study of steady torsional motion of a circular cylinder, neglecting the convective acceleration. For the resulting linearized theory he obtains a solution in terms of exponentials in time and Bessel functions of the radial distance. In particular, he finds the longitudinal extension. As $t\rightarrow\infty$, a certain permanent extension results. Using trigonometrical series, he obtains a similar result for a shaft of rectangular section. He compares the results with those from classical linear elasticity, and he mentions certain gross experimental measurements. [His remarks regarding the work of Poynting indicate that he is unaware of the exact solution of the former problem in the classical theory of finite strain; cf., e.g., §42H of the reviewer's paper cited above. Of course the elastic solution does not predict permanent set. The author's analysis, while incomplete, appears to be the first effort to study the visco-elastic theory of extension induced by torsion.]

C. Truesdell (Bloomington, Ind.).

Sawaragi, Yoshikazu, and Tokumaru, Hidekatsu. On fundamental equation of the dynamical behaviours of nonlinear visco-elastic bodies. Mem. Fac. Eng. Kyoto Univ. 16, 100–111 (1954).

The authors consider one-dimensional models of visco-elastic bodies. They are continuous analoga of systems composed not only of the usual springs and dashpots [see for instance ter Haar, Physica 16, 719–737, 738–752 (1950); these Rev. 13, 406] but also of solid friction mechanisms. These models show certain hysteresis and non-linear viscosity effects.

W. Noll (Berlin).

Knopoff, Leon. On the dissipative viscoelastic constants of higher order. J. Acoust. Soc. Amer. 26, 183–186 (1954).

In a purely formal way the author writes down a linear formula for the stress in terms of the linearized strain and its pth derivative and discusses the solutions which are exponential in the time. He concludes that in order for the process described by these equations to be dissipative it is necessary that p be odd and that an inequality analogous to that of Stokes-Duhem (the author's date 1851 should be replaced by 1901) for the ordinary viscosities shall hold. [The author is apparently unaware of the difficulties surrounding this problem and of the enormous literature on visco-elasticity. Cf., e.g., the citations in the footnotes to §81 of the reviewer's paper, J. Rational Mech. Anal. 1, 125–171, 173–300 (1952); 2, 593–616 (1953); and the remarks on pp. 671–674, 681–682 of the reviewer's paper, ibid. 2, 643–741 (1953); these Rev. 13, 794; 15, 178, 757.]

C. Truesdell (Bloomington, Ind.).

Franciosi, Vincenzo. Le aste sottili pressoinflesse in regime viscoso. Rend. Accad. Sci. Fis. Mat. Napoli (4) 19 (1952), 57–63 (1953).

Die Arbeit behandelt einen durch Transversal- und Normallasten beanspruchten dünnen Stab aus einem Material mit elastischer Nachwirkung. Sie verallgemeinert eine Untersuchung von G. Krall [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 2, 371–377 (1947)] auf den Fall, dass der Nachwirkungskern nicht nur von der Zeit t, sondern auch vom Ort x abhängt und dass er die spezielle Form $\Phi(t,x)=\varphi(t)f(x)$ hat. Für die (als sehr klein vorausgesetzte) Durchbiegung $w(x,t)$ ist eine Integro-Differentialgleichung massgebend. Nach der Methode von Krall wird für $\omega(x,t)=(\partial/\partial x)w(x,t)$ eine Entwicklung nach den Eigenfunktionen einer gewissen linearen homogenen Integralgleichung angesetzt. Indem der Verfasser nur die ersten m Glieder dieser Entwicklung berücksichtigt, gelangt er zu einem System von gewöhnlichen Differentialgleichungen. Für den Spezialfall $\varphi(t)=\exp(-t)$ und zeitunabhängiger Lasten wird eine explizite Lösung angegeben.

W. Noll (Berlin).

Bullough, R., and Bilby, B. A. Uniformly moving dislocations in anisotropic media. Proc. Phys. Soc. Sect. B. 67, 615–624 (1954).

Der Ausgangspunkt der Untersuchung ist die sich auf ein anisotropes elastisches Medium beziehende Bewegungsgleichung

$$(1)\qquad c_{ijkl}\frac{\partial^2 u_k}{\partial x_j\partial x_l}=\rho\frac{\partial^2 u_i}{\partial t^2}$$

in der die u_k die Verrückungen entlang der Achsen eines Descartesschen Koordinatensystems und die c_{ijkl} die Komponenten des elastischen Tensors sind. ρ bedeutet die Dichte. Die Versetzungslinie wird parallel zur x_3-Richtung angenommen und ihre Bewegung (mit der Geschwindigkeit c) parallel zur x_1-Achse. Die gestellte Aufgabe ist, eine Lösung von (1) zu finden, welche Störungen beschreibt, die sich in der $x_2=0$ Ebene fortpflanzen und verschwinden wenn $|x_2|\rightarrow\infty$. Die allgemeine Form dieser Lösung ist

$$(2)\qquad u_j(x_1',x_2)=\sum_n C_n P_{jn}\exp\{+s(-\lambda_n x_2+ix_1')\},$$

wo $n=1, 2, 3$ ist, die P_{jn} und λ_n Funktionen der elastischen Konstanten, C eine willkürliche komplexe Konstante und $x_1'=x_1-ct$ ist. Das Problem wird auf den Fall spezialisiert, dass (in der gewohnten Bezeichnungsweise) die elastischen Konstanten c_{14}, c_{15}, c_{24}, c_{25}, c_{34}, c_{35}, c_{46} und c_{56} verschwinden. Das System (1) zerfällt dann in zwei voneinander unabhängige Gleichungen, die dann für sich in Bewegung befindenen Stufen- und Schraubenversetzungen gelöst werden. Im ersteren Fall ist der Burgerssche Vektor entlang der x_1-Richtung, im letzteren entlang der x_3-Richtung orientiert. Spezialisiert man die erhaltenen Resultate für den statischen bzw. isotropen Fall, so gehen sie in die Resultate von J. D. Eshelby [Proc. Phys. Soc. Sect. A. 62, 307–314 (1949); Philos. Mag. (7) 40, 903–912 (1949)] über. Die Rechnungen werden noch auf das Versetzungsmodell von Peierls und Nabarro erweitert und im letzten Paragraphen werden die Resultate auf die hexagonal am dichtesten gepackten Metalle Mg, Zn und Cd angewendet und die erhaltenen Ergebnisse graphisch dargestellt.

T. Neugebauer (Budapest).

MATHEMATICAL PHYSICS

Finzi, Bruno. La fisica matematica. Discorso inaugurale. Ist. Lombardo Sci. Lett. Rend. Parte Generale e Atti Ufficiali (3) 17(86), 86–98 (1953).

Bastin, E. W., and Kilmister, C. W. Eddington's theory in terms of the concept of order. Proc. Cambridge Philos. Soc. 50, 439–448 (1954).

This paper examines connections between Eddington's work as propounded in his book "Fundamental theory" [Cambridge, 1946; these Rev. 11, 144] and the theory of a previous paper by the authors [same Proc. 50, 278–286 (1954); these Rev. 15, 760]. *A. H. Taub.*

Drobot, S. On the foundations of dimensional analysis. Studia Math. 14, 84–99 (1953).

A paradox-free exposition of dimensional analysis is given, in which dimensional analysis is considered as the study of linear dependence relations in a real vector space. Thus power-products $L^\alpha M^\beta T^\gamma\cdots$ are construed isomorphically as linear combinations $\alpha L+\beta M+\gamma T\cdots$. An application to sampling theory is sketched [cf. S. Drobot and M. Warmus, Rozprawy Mat. 5, (1954); these Rev. 16, 55]. *G. Birkhoff* (Cambridge, Mass.).

Optics, Electromagnetic Theory

Fletcher, A., Murphy, T., and Young, A. Solutions of two optical problems. Proc. Roy. Soc. London. Ser. A. 223, 216–225 (1954).

A solution is given for the problem of finding the spherically symmetrical refractive index function within a solid sphere such that light from an external point A and passing through the sphere will come to a focus at a second external point B. This problem was solved by Luneberg [Mathematical theory of optics, Brown Univ., Providence, R. I., 1944; these Rev. 6, 107], but is here solved in a somewhat different fashion and is developed in a form for numerical computation.

The second problem solved is that of a lens with a front surface plane and the back surface in the form of a circular cylinder, the problem being to find the radially symmetrical refractive index function in the lens so that a parallel beam, perpendicular to the plane face will converge to a point after passing through the lens.

Both problems lead to integral equations of Abel's type. The solution of the second problem is given as sech $(\pi r/2F)$ where r denotes the distance from the axis of the cylinder and F the distance of the focus from the plane face. *M. Herzberger* (Rochester, N. Y.).

Suzuki, Tatsuro. Design of a lens systems having aspherical surfaces. I. Tech. Rep. Osaka Univ. 3, 215–223 (1953).

The differential method of correcting lens aberrations is applied to show how spherical surfaces in the lens can be deformed to reduce the aberrations. A numerical example illustrates how a small deformation of one lens element can produce a marked reduction in spherical aberration. *M. Herzberger* (Rochester, N. Y.).

Gautier, Pierre. Calcul numérique des trajectoires dans les systèmes centrés de l'optique électronique. J. Phys. Radium (8) 14, 524–532 (1953).

In this paper the author gives a method of obtaining rapid numerical calculation of electron paths in a rotational symmetric electron field based on finite-difference calculus. This method is developed for second-order differential equations which do not contain first-order derivatives, although the general case can be reduced to this by a simple transformation. The paraxial trajectories of electrons for the case of a magnetic field of bell-shaped form have been calculated and the values of the focal distance, the intersection point of the trajectory with the axis of the system, and the spherical aberration constant are found to be in agreement to within 0.2% of the exact values obtained by Glaser [Z. Physik 117, 285–315 (1941); these Rev. 4, 32]. The method is extended to more general systems (third-order aberrations) by considering the non-linear part of the differential equation as perturbation terms of the paraxial equation. The author discusses briefly the case of electrostatic fields in combination with space charge and that of chromatic aberration. *N. Chako* (New York, N. Y.).

Lansraux, Guy. Conditions fonctionnelles de la diffraction instrumentale. Cas particulier des zéros d'amplitude de figures de diffraction de révolution. Cahiers de Physique no. 45, 29–39 (1953).

It is well known that the diffraction pattern of an object point formed by an optical system on an image plane $z=z_1$, can be represented by a double Fourier integral of the form

$$(1)\quad F(x,y,z_1)=-\frac{ik}{2\pi}\int\int_D G(p,q,z_1)\exp(ik(xp+yq))\,dpdq,$$
$$g(p,q,z_1)=f(p,q)\exp(ik(W+sz))\cdot s^{-1},$$

where p, q, s are optical direction cosines in the image space, f is the light distribution over the domain D (the aperture of the exit-pupil), and W is the Hamilton mixed characteristic of the system. For rotational symmetric systems F and G form an Hankel transform with respect to the Fourier kernel $J_0(2\pi r\rho)$ $(r=(x^2+y^2)^{1/2},\ \rho=(p^2+q^2)^{1/2})$ provided only spherical aberration is present in the system. In this paper the author discusses briefly the analytical properties (without proofs) of F and G for domains D bounded by analytic curves, in particular concentric circles. To calculate F the author makes use of Weierstrass' theorem by approximating G by polynomials with argument $(1-\rho^2)$ in each of the subdivided regions D_i of D (D=unit circle). Finally, the author considers the problem where $F(r)$ vanishes at certain distances from the axis, i.e.

$$F(r_1)=F(r_2)=\cdots=F(r_k)=0,$$

and the question is to find $G(\rho)$ for a system free of aberrations and corrected at the focal plane, that is, $G=f$. By approximating G by polynomials $P_i(1-\rho^2)$ with real coefficients, the author sets up a system of n equations (inhomogeneous) from which the coefficients are evaluated. [Reviewer's remarks. The analytical properties of F and G with a more general Fourier kernel are given in I. N. Sneddon, "Fourier transforms" [McGraw-Hill, New York, 1951; these Rev. 13, 29]. It should also be pointed out that the approximation of G by polynomials with argument $(1-\rho^2)$ does not give a sufficiently accurate value of F even for small aberration (spherical) if r lies outside the geometrical shadow.] *N. Chako* (New York, N. Y.).

Stařček, Imrich. A plane light wave in a totally anisotropic nonconducting medium. Mat.-Fyz. Sbornik Slovensk. Akad. Vied Umení 1, 18–30 (1951). (Slovak. Russian and French summaries)

In a previous work ["A plane electromagnetic wave in a medium which is electrically or magnetically anisotropic" (in Slovak), Bratislava, 1946, (not accessible to the reviewer)] the author has derived an equation (which he reproduces) for the possible velocities of a plane wave in an anisotropic medium, assuming the tensors $\bar{\epsilon}$ and $\bar{\mu}$ to have the same principal axes. He now gives the corresponding, rather involved calculations for general $\bar{\epsilon}$ and $\bar{\mu}$.

F. V. Atkinson (Ibadan).

Grinberg, G. A., Lebedev, N. N., Skal'skaya, I. P., and Uflyand, Ya. S. Wave problem for a parabolic mirror. Doklady Akad. Nauk SSSR (N.S.) 95, 961–963 (1954). (Russian)

A critical survey of papers on the diffraction of electromagnetic waves by a parabolic cylinder, viz., [1] P. S. Epstein, Dissertation, Munich, 1914; [2] P. S. Epstein, Enzyk. Math. Wiss., Bd. 5_3, Teubner, Leipzig, 1915, p. 511; [3] W. Magnus, Jber. Deutsch. Math. Verein. **50**, 140–161 (1940); [4] W. Magnus, Z. Physik **118**, 343–356 (1941) [these Rev. **2**, 56; **9**, 125]. Limitations of the method used in [1] and [2] are mentioned (convergence of the series in part of space only). Lack of a detailed proof in [3] for the fact that the solution satisfies the radiation condition is emphasized and a full proof is announced. An expansion in terms of products of parabolic cylinder functions of integral (positive and negative) order is derived for the field produced by a line source in the focal line of the cylinder. The proof is sketched and based on [3]. The result is used for filling a gap in [4] by showing that the diffracted wave in [4] satisfies the condition of being finite on the focal line. The series for the diffracted wave found by the authors is shown to converge only in a finite part of the interior of the cylinder.

W. Magnus (New York, N. Y.).

Imai, Isao. Die Beugung elektromagnetischer Wellen an einem Kreiszylinder. Z. Physik 137, 31–48 (1954).

The author starts with the Fourier-series solution of the problem of diffraction of a plane electromagnetic wave by a large perfectly conducting cylinder. This series is then converted into a contour integral in the complex plane. This contour integral is then evaluated approximately in very much the same manner as in the similar case of diffraction by a large sphere [G. N. Watson, Proc. Roy. Soc. London. Ser. A. **95**, 83–99 (1918); H. Bremmer, Terrestrial radio waves, Elsevier, New York, 1949; these Rev. **11**, 295]. In the lit region the saddle point contribution can be interpreted in terms of geometrical optics, while in the shadow region the residue series representation is appropriate, each term corresponding to a "creep-wave", a mode propagated around the cylinder. These results are shown to be in agreement with those obtained by a less rigorous procedure by Franz and Deppermann [Ann. Physik (6) **10**, 361–373 (1952); these Rev. **14**, 518], except that one term in the Franz-Deppermann solution is shown to be spurious. A similar treatment of the problem is contained in a paper by F. G. Friedlander [these Rev. **16**, 87] dealing with a related subject.

J. Shmoys (New York, N. Y.).

Deppermann, K., und Franz, W. Theorie der Beugung an der Kugel unter Berücksichtigung der Kriechwelle. Ann. Physik (6) 14, 253–264 (1954).

The authors extend their work on the creeping wave [Ann. Physik (6) **10**, 361–373 (1952); these Rev. **14**, 518] to the sphere. Unlike the cylinder, the sphere exhibits singularities in the phase function in the neighborhood of the pole. The theory predicts the appearance of the interference effects found in the case of spheres of diameters of the order of one hundred wavelengths. Curves are given of the amplitude behavior of the creeping wave.

W. K. Saunders (Washington, D. C.).

Oberhettinger, F. Diffraction of waves by a wedge. Comm. Pure Appl. Math. 7, 551–563 (1954).

The various two-dimensional problems concerning the diffraction of "monochromatic" waves, which may be sound waves or electromagnetic waves, depend on finding the Green's functions G_1 and G_2 of the wave equation $\Delta u+k^2u=0$; G_1 satisfies the boundary condition $G_1=0$, G_2 the boundary condition $\partial G_2/\partial N=0$.

In the case of diffraction by a wedge, the region in question is an angular region $0\leqq\phi\leqq\alpha$ in cylindrical coordinates. The Green's function has to behave near the singularity (ρ', ϕ') like

$$\Phi^{(i)}=\frac{2i}{\pi}K_0(\gamma r),$$

where K_0 is the modified Hankel function, $\gamma=ik$ for convenience and $r^2=\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')$, and satisfy the prescribed boundary conditions. This means that the Green's function is of the form $\Phi^{(i)}+\Phi^{(r)}$ where $\Phi^{(r)}$ has no singularities in $0\leqq\phi\leqq\alpha$.

The author claims that all previous determinations of these Green's functions have been more or less involved, and that they can be easily found by conventional methods. The essential point is that

$$\Phi^{(i)}=\frac{4i}{\pi^2}\int_0^\infty K_{i\lambda}(\gamma\rho)K_{i\lambda}(\gamma\rho')\cos[\lambda(\pi-|\phi-\phi'|)]d\lambda.$$

Then if one takes

$$\Phi^{(r)}=\frac{4i}{\pi^2}\int_0^\infty K_{i\lambda}(\gamma\rho)K_{i\lambda}(\gamma\rho')[f_1(\lambda)e^{\lambda\phi}+f_2(\lambda)e^{-\lambda\phi}]d\lambda,$$

the functions $f_1(\lambda)$ and $f_2(\lambda)$ are readily determined from the boundary conditions. The author's idea of "easy" is not mine; the analysis is admittedly straightforward, but it involves a fairly extensive knowledge of the properties of Bessel functions; and considerable manipulative skill is needed in deducing the more well known forms of these Green's functions.

The corresponding results for three dimensions (with a point source instead of a line source) are deduced from the two-dimensional results by a device due to Weyl.

E. T. Copson (St. Andrews).

Reiche, Fritz. On diffraction by an infinite grating. Division of Electromagnetic Research, Institute of Mathematical Sciences, New York University, Research Rep. No. EM-61, i+13 pp. (1953).

"We consider the diffraction of a plane wave by an infinite grating of identical circular cylinders. The total scattered field can be represented as a series which is the sum of waves scattered by the individual cylinders. We shall not be concerned with the coefficients in the representations of

the waves, but rather consider only the summation of the waves over the individual cylinders of the grating. We show that the scattered cylindrical waves combine to form a finite number of plane waves propagated in the directions given by the well-known grating diffraction formula, and in addition an infinite number of "surface waves" propagated along the grating with amplitudes which decrease exponentially with the distance from the grating." (From the author's summary.) *E. T. Copson* (St. Andrews).

Baldwin, George L., and Heins, Albert E. On the diffraction of a plane wave by an infinite plane grating. Math. Scand. **2**, 103–118 (1954).

Consider a plane monochromatic electromagnetic wave of arbitrary polarization which is incident normally on a diffraction grating in the plane $x=0$, in rectangular cartesian coordinates. This grating consists of an infinite set of identical perfectly conducting strips parallel to the z-axis with the spacing between any two adjacent strips equal to the width of one of them. This problem is formulated as an integral equation which can be solved by the Wiener-Hopf technique; and the authors find the amplitude, phase and direction of propagation of the transmitted and reflected waves at a great distance. The problem is, admittedly, rather special; but, with the particular spacing chosen, it is possible to find the reflection and transmission characteristics in closed form without assuming that the ratio of the aperture width to the free-space wave-length is small.

The problem is, in fact, equivalent to that of determining the effect of a conducting fin inserted in a parallel plane wave guide which is excited by a dominant mode, the fin being perpendicular to the direction of propagation and extending from one conducting plane half way to the other. It is this equivalent problem which is formulated as an integral equation. *E. T. Copson* (St. Andrews).

Försterling, K. Reflexion und Brechung elektromagnetischer Wellen an einem geschichteten Medium. Hochfrequenztech. Elektroak. **63**, 112–116 (1954).

The author deals with the propagation of plane electromagnetic waves in an inhomogeneous (but slowly varying) anisotropic medium. He goes beyond the usual phase integral approximation in that he calculates the first order reflection and coupling terms. Similar calculations have been made, for nth order terms by Bremmer [Comm. Pure Appl. Math. **4**, 105–115 (1951); these Rev. **13**, 462], Landauer [Physical Rev. (2) **82**, 80–83 (1951); these Rev. **13**, 287] and H. B. Keller and J. B. Keller [Mathematics Research Group, Washington Square College of Arts and Science, New York University, Research Rep. No. **EM-33** (1951); also H. B. Keller, ibid. Nos. **EM-56, EM-57** (1953); these Rev. **13**, 346; **15**, 585].

Although references to three papers, numbered [1], [2], and [4] are to be found in the text, no bibliography is given. *J. Shmoys* (New York, N. Y.).

De Socio, Marialuisa. Sulla rappresentazione del campo elettromagnetico in una guida d'onda a pareti assorbenti. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **16**, 63–68 (1954).

It was first pointed out by Brillouin that when a TM wave is propagated along a parallel-plate wave guide with perfectly conducting walls, the effect is the same as the superposition of a plane wave and the waves resulting from it by reflection at the walls. The present paper is concerned with the extension of this result to the case when the walls are not perfectly conducting. More precisely, using cartesian coordinates, the walls of the guide are the half-spaces $y>d$ and $y<-d$, each filled with a medium of finite conductivity; and the region $-d<y<d$ is filled with a homogeneous dielectric. The analysis is quite straight-forward, but leads to a complicated transcendental equation, which can only be solved approximately. *E. T. Copson.*

Tabellini, Paola. Su una guida d'onda con uno schermo conduttore. Atti Sem. Mat. Fis. Univ. Modena **6** (1951–52), 87–97 (1953).

This paper is concerned with the propagation of monochromatic electromagnetic waves in the interior of a perfectly conducting circular cylindrical wave guide into which has been inserted an imperfectly conducting coaxial screen. If we use the obvious cylindrical coordinates (ρ, θ, z), the space inside the guide is divided into two regions, the region 1, for which $0<\rho<l$, inside the screen and the region 2, for which $l<\rho<b$, between the screen and the guide. The problem is then to find solutions of Maxwell's equations in which $\mathbf{E}$ and $\mathbf{H}$, the electric and magnetic vectors, are proportional to $e^{i\omega(t-z/V)}$; these vectors satisfy the usual boundary conditions on the guide $\rho=b$ and the following conditions on the screen: (i) the tangential component $\mathbf{E}_t$ of $\mathbf{E}$ is continuous across $\rho=l$; (ii) $\mathbf{H}$ is discontinuous across $\rho=l$, where $(\mathbf{H}_2-\mathbf{H}_1)\wedge\mathbf{n}=-\gamma h\mathbf{E}_t$; here $\mathbf{n}$ is the unit vector normal to the screen drawn outwards, γ is the conductivity of the screen and h its very small thickness. It turns out that the only symmetrical modes are transverse electric or transverse magnetic, and that the unsymmetrical modes are, in general, hybrid. *E. T. Copson* (St. Andrews).

Gincburg, M. A. A gyrotropic wave guide. Doklady Akad. Nauk SSSR (N.S.) **95**, 489–492 (1954). (Russian)

The author first considers the propagation of electromagnetic waves in a circular guide of infinite length completely filled with a "gyrotropic" medium, i.e., a medium which is simultaneously gyroelectric and gyromagnetic. He finds that the waves undergo a Faraday rotation and that the configuration of the electromagnetic field is such that purely E-type and purely H-type waves cannot exist but rather a mixture of the two is present. [See H. Gamo, J. Phys. Soc. Japan **8**, 176–182 (1953); these Rev. **14**, 823; H. Suhl and L. Walker, Physical Rev. (2) **86**, 122–123 (1952).] He extends these deductions to the case where the guide is shorted at both ends by perfectly conducting plates, i.e. the resonator case.

He then computes an explicit expression for the Faraday rotation in terms of the guide radius and the components of the dielectric and permeability tensors under the simplifying restriction that the rotational components of the tensors are small enough to be considered as perturbations. He also suggests the possibility of determining the Faraday rotation by the " variational method of Ritz and Galerkin" (the reviewer presumes the author is referring to what is often called the "Ritz method"). Lastly he treats the case where the center conductor of a coaxial line is a gyrotropic substance. [See A. A. T. M. van Trier, Appl. Sci. Research B. **3**, 305–371 (1953).] *C. H. Papas* (Pasadena, Calif.).

Grinberg, G. A., and Bonštedt, B. È. Elements of an exact theory of the wave field of transmission lines. Akad. Nauk SSSR. Žurnal Tehn. Fiz. **24**, 67–95 (1954). (Russian)

The author treats the problem of propagation of electromagnetic waves along a circularly cylindrical wire parallel

to the (plane) earth's surface. The dielectric constant and conductivity of the wire and of the earth are arbitrary. It is immediately assumed, however, that the radius of the wire is much smaller than its elevation above the earth; the effect of the earth is taken as a perturbation and only the first order effect is included. In this manner the effective parameters (series impedance and shunt admittance per unit length) of this transmission line operating in the dominant mode are calculated. The calculation is repeated with the earth, instead of being represented by a semi-infinite medium of large refractive index, introduced into the problem through skin-effect type boundary conditions at its surface. The numerical results of the simplified calculation agree well with those of the more rigorous one.

J. Shmoys (New York, N. Y.).

***Wagner, Karl Willy. Elektromagnetische Wellen. Eine Einführung in die Theorie als Grundlage für ihre Anwendung in der elektrischen Übertragungstechnik.** Verlag Birkhäuser, Basel-Stuttgart, 1953. 267 pp. DM 33.30.

The volume is an intermediate-level text written for electrical engineers. The exposition is very lucid and complete. Topics, such as the concept of group velocity, are handled with great pedagogical skill. A background of electrostatics is assumed and the book builds from this under the chapter headings: (1) Plane waves, (2) Reflection and refraction, (3) Waves along wires, (4) Transmission lines, (5) Wave guides, (6) Dipole antennas, (7) The ionosphere. The material in chapters 3, 4, and 7, which lies within the author's field of interest, is particularly complete. A bit of the material in chapters 5 and 6 is somewhat out of date. For example, the treatment of the half-wave dipole antenna omits not only all recent work, but even the classic result of P. S. Carter [Proc. I. R. E. **20**, 1004–1041 (1932)]. This type of omission is not particularly unfortunate since the book gives the student a sound foundation in the fundamentals of electromagnetic theory, one which should permit him to grasp the newer material available in the literature. The work, which is excellently printed and well illustrated, should be a valuable addition to the limited modern engineering literature available in the German language. *W. K. Saunders* (Washington, D. C.).

On order

***Maxwell, James Clerk. A treatise on electricity and magnetism. 3d ed.** Two volumes bound as one. Vol. I, xxxii+506 pp. +13 plates; vol. II, xxiv+500 pp. +7 plates. Dover Publications, Inc., New York, 1954. $4.95.

Reproduction by photo-offset of the third edition [2 vols., Cambridge, 1891].

***Durand, E. Electrostatique et magnétostatique.** Masson et Cie, Paris, 1953. xii+774 pp. 5760 fr.

In the release accompanying this book the publishers state that it is intended, among others, for teachers in (French) secondary schools; lucky are the pupils in a high school where the science teachers have this kind of a background.

Durand's book is the most complete and up-to-date treatise on electrostatics and magnetostatics (i.e. on all phenomena in electricity and magnetism which are invariant with time) this reviewer has encountered. In addition to the topics usually covered in texts on this subject, such as fundamental equations of electrostatics and magnetostatics, charge distributions, conductors, potential distributions and solutions of Laplace's equation, current distributions, etc., it contains chapters on molecular structure, thermodynamics, theory of elasticity, and crystal structure, as applied to electricity and magnetism. While it is debatable whether the special topics should be included in a general treatise on electricity and magnetism, as long as no other single text appears to cover them, it is convenient to have them available for ready reference.

The first chapter reviews very briefly some of the basic mathematical formulas and techniques used. This review, so brief as to be little more than a compendium of formulas, indicates the extent of the mathematical preparation expected from the reader. Given, however, the necessary background, a study of the book will prove most rewarding both to the graduate student seeking an understanding and general grasp of the subject and to the scientist interested in precise technical information on a particular point.

While no exercises are provided to be worked out by the reader, numerous problems, which are merely stated in other books, are stated in detail in the text. This is particularly true of Ch. IV on conductors, Ch. XIV on given current distributions, and Ch. X, XI, and XII dealing with analytical and numerical solutions of Laplace's equation. These last three chapters are invaluable to the electron-tube designer. Indications are also given on a possible approach to more difficult problems. The bibliography appears to be up-to-date, and references are included to original papers, particularly in the sections dealing with the solutions of problems.

In view of the general excellence of this text, it would be highly desirable to make it more readily available to the American student. An English translation using a notation more common in this country and an amplified mathematical introduction would fulfill this purpose. In order to restrict the material to what could be covered in a year's course, the special topics could be omitted and issued (possibly amplified) in a separate volume. A graduate course in electricity and magnetism using Durand's book as a text should prove a wonderful experience to teacher and students.

J. E. Rosenthal (Passaic, N. J.).

Thomas, T. S. E. The capacitance of an anchor ring. Australian J. Physics **7**, 347–350 (1954).

The author has computed the capacity of an anchor ring of radius r and radius of generating circle R, completing a problem begun by W. M. Hicks [Philos. Trans. Roy. Soc. London **172**, 609–652 (1882)]. The formula, a convergent infinite series involving elliptic integrals, can be evaluated for $r<R$. The limiting case $r=R$ is easily obtained by inversion. A table of capacities is given for r/R from 0.05 to 1.00.

W. K. Saunders (Washington, D. C.).

Muller, D. E. Boolean algebras in electric circuit design. Proceedings of the symposium on special topics in applied mathematics, Northwestern University, 1953. Amer. Math. Monthly **61**, no. 7, part II, 27–28 (1954).

***Gross, B. The circuit function—a new concept in electrical network theory (preliminary note).** New research techniques in physics, pp. 373–378. Symposium organized by the Academia Brasileira de Ciências and Centro de Cooperación Científica para America Latina (UNESCO) under the auspices of the Conselho Nacional de Pesquisas do Brasil, Rio de Janeiro and São Paulo, July 15–29, 1952. Rio de Janeiro, 1954.

Generalizations from lumped, finite, reactive networks to lossless continuous parameter systems are treated. A function of frequency corresponding to the amplitude factor of

the indicial admittance is defined whereby the indicial admittance of continuous systems is expressed as the Fourier transform of the defined circuit function. Some familiar results, such as Carson's infinite integral are then derived in terms of either the indicial admittance or the circuit function. *R. Kahal* (Monterey, Calif.).

Quantum Mechanics

Falk, Gottfried. Die Struktur des Grössenbereiches von klassischer Mechanik und Quantenmechanik. Z. Physik **135**, 431–472 (1953).

This gives a relatively simple axiomatic approach to the correspondence principle along purely formal lines, within the framework of 'modern algebra'. The observables in both classical and quantum theory are treated as elements of a Lie algebra of what are essentially infinite formal series, certain automorphisms of these algebras corresponding to motions of the systems. Formally there is some similarity with work of van Hove [see, e.g., Acad. Roy. Belgique. Bull. Cl. Sci. (5) **37**, 610–620 (1951); these Rev. **13**, 519], who however used a Hilbert space framework but did not examine the relativistic case or include spin in his systems, as the present author does. The present work, which treats only the case of a single particle, is a synthesis of new and older work of the author [Z. Physik **130**, 51–68 (1951); **131**, 470–480 (1952); **132**, 44–53 (1952); Math. Ann. **123**, 379–391 (1951); these Rev. **13**, 96, 946; **14**, 1045; **13**, 426].
I. E. Segal (Princeton, N. J.).

De Donder, Th. A toute mécanique ondulatoire correspond une mécanique statistique. II. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **38**, 1125–1128 (1952).

[For part I see same Bull. (5) **27**, 438–440 (1941); these Rev. **3**, 320.] The author points out already known formal analogies between quantum mechanics and classical statistical mechanics [cf., e.g., Blochinzew, Grundlagen der Quantenmechanik, Deutscher Verlag der Wissenschaften, Berlin, 1953, §44 and the papers cited there]. *J. Werle.*

Rosenstock, Herbert B. Quantization of a nonlinear field theory. Physical Rev. (2) **93**, 331–333 (1954).

The paper deals with a non-linear field equation for a function ψ which, in the classical case, has spherically-symmetric static solutions describing a field concentrated in a small region of space that can be regarded as a particle. In order to quantize ψ, the author writes it as a linear function of such spherically-symmetric particle solutions, each with its center at a different lattice point of a cubic lattice with a fixed lattice constant D, the coefficients in this expansion being taken as operators satisfying with their conjugate momenta the usual commutation relations. The Hamiltonian is set up, and approximate expressions for the energy levels of the system are obtained provided that D is sufficiently large and one of the constants in the field equation sufficiently small. The problem of the non-linear field is a difficult one, and there is still much to be done in this area. *N. Rosen* (Haifa).

Gel'fand, I. M., and Minlos, R. A. Solution of the quantum field equations. Doklady Akad. Nauk SSSR (N.S.) **97**, 209–212 (1954). (Russian)

A method is proposed for finding explicit solutions to the equations of quantum field theory. As an illustration, a single scalar field ψ satisfying the non-linear equation

$$\Box\psi-k^2\psi=\lambda\psi^2+J$$

is considered, J being an external c-number source density. The quantity to be computed is $z=(\langle\exp(-iL)\rangle_0)^{-1}$, where L is the Lagrangian operator integrated over the whole space-time. All scattering matrix elements etc. can be expressed as derivatives of z with respect to J. The explicit formula for z is

$$z=c\int dt_1\cdots dt_n \exp\left[\omega\left(\sum t_kJ_k-\tfrac{1}{3}\lambda\sum t_k^3+\tfrac{1}{2}i\sum\sum R_{kl}t_lt_k\right)\right],$$

where ω is a 4-dimensional volume, the R_{lk} are numerical coefficients, and the t_k are integration variables defined one at each point of a lattice filling the volume ω. The value of z is to be obtained by a limiting process as the lattice-points become more and more numerous and close together. The authors say they learned only after writing this paper that an essentially identical method was published by R. P. Feynman [Physical Rev. (2) **80**, 440–457 (1950); these Rev. **12**, 889]. *F. J. Dyson* (Princeton, N. J.).

Deprit, André. Problèmes de Cauchy en théorie quantique des champs. Ann. Soc. Sci. Bruxelles. Sér. I. **68**, 119–132 (1954).

This is devoted to the formulation, in terms of the theory of distributions, of an analog to Feynman's Green's-function treatment of the Schrödinger equation. *I. E. Segal.*

Ioffe, B. L. Systems of covariant equations in the theory of quantum fields. Doklady Akad. Nauk SSSR (N.S.) **95**, 761–764 (1954). (Russian)

Considering as a convenient example the theory of Dirac nucleons interacting with a neutral pseudoscalar meson field, the author outlines a simple and general method of obtaining the covariant integral equations relating the many-particle Green's functions in any quantum field theory. Let $G(x,x')$ be the one-nucleon Green's function in the presence of an external nucleon source-density $J(x)$; this satisfies the Schwinger functional derivative equation [Proc. Nat. Acad. Sci. U. S. A. **37**, 452–455, 455–459 (1951); these Rev. **13**, 530]

$$(1)\quad [i\gamma_\mu(\partial/\partial x_\mu)-m+ig\gamma_5\langle\phi(x)\rangle + ig\gamma_5(\delta/\delta J(x))]G(x,x')=\delta(x-x').$$

The one-nucleon n-meson Green's function G_n in the absence of the external source is defined by expanding the functional $G(x,x')$ as a Volterra series in powers of the function $J(x)$:

$$(2)\quad G(x,x')=\sum_{n=0}^{\infty}\int G_n(x,x',x_1,\cdots,x_n) \times J(x_1)\cdots J(x_n)dx_1\cdots dx_n.$$

The integral equations relating the various G_n to one another are obtained immediately by substituting (2) into (1). This method of deriving the equations is due essentially to R. Utiyama, S. Sunakawa and T. Imamura [Progress Theoret. Physics **8**, 77–110 (1952); these Rev. **15**, 82].
F. J. Dyson (Princeton, N. J.).

Demeur, Marcel. Etude de l'intéraction entre le champ propre d'une particule et un champ électro-magnétique homogène et constant. Acad. Roy. Belgique. Cl. Sci. Mém. Coll. in 8° **28**, no. 5, 98 pp. (1953).

The method of Géhéniau [Physica **16**, 822–830 (1950); these Rev. **12**, 872] is employed to obtain radiative corrections to the self-energy of an electron or nucleon in a constant non-quantized electric field. Like Furry [Physical Rev. (2) **81**, 115–124 (1951); these Rev. **12**, 785], Demeur uses the bound-state representation but obtains a closed expression for the corrections valid for arbitrary values L of the constant field. From this he easily derives corrections involving the lowest powers of L, obtaining results in agreement with Borowitz and Kohn [ibid. **76**, 818–827 (1949)] and Luttinger [Helvetica Phys. Acta **21**, 483–496 (1948)]. The central new result of the paper which is of considerable mathematical interest is a representation of the propagators Δ and D for the free-field and for the constant external field as a single integral. [Reviewer's remark: In obtaining the formula for the propagator the author waves the magic wand "quantisation" over Weber's differential equation to restrict the wave functions to a discrete set of solutions. If space is infinite, as is assumed, it is not apparent to the reviewer why this restriction is justified. If the well-known "large box" trick is used, then with a constant field the boundary conditions on the walls of the box surely become important.] *A. J. Coleman* (Toronto, Ont.).

Van Kampen, N. G. The symmetry relation of the S matrix in the complex plane. Physica **20**, 115–123 (1954).

In his previous discussion [Physical Rev. (2) **91**, 1267–1276 (1953); these Rev. **15**, 588] of the properties of the scattering matrix $S(p)$ in the non-relativistic spherically symmetric case, the author was unable to prove the condition $S(-p)=S^*(p)$ which is known to obtain for potential scattering. He here shows that this condition follows from the additional assumption that the total energy has a lower bound. *A. J. Coleman* (Toronto, Ont.).

Cap, Ferdinand. Zur Kopplung eines Dirac-Feldes mit Bosonen vom Spin 1. Acta Physica Austriaca **8**, 191–197 (1953).

Conservation theorems are derived and interpreted for the tensor-densities constructed from spinors, and further used to determine which types of coupling between particle fields of spin $\frac{1}{2}$ and 1 are admissible. The differences between the results obtained for vector and pseudovector fields are discussed. *E. Gora* (Providence, R. I.).

Freese, Ernst. Gebundene Teilchen und Streuprobleme in der Quantenfeldtheorie. Z. Naturforschung **8a**, 776–790 (1953).

This paper introduces a new definition of the wave functions associated with any quantum mechanical system and derives the equations which they satisfy. The various recent attempts (by Levy, Bethe-Salpeter, Tamm-Dancoff et al.) to give a relativistic treatment of systems of more than one particle would appear to be various approximations to this theory.

In order to calculate the T-products in the S-matrix, Wick [Physical Rev. (2) **80**, 268–272 (1950); these Rev. **12**, 380] introduced the more easily calculable S-product. Whereas the T-product has a relativistically invariant meaning in the Heisenberg representation, the S-product does not. The present author turns Wick's procedure around and having expressed the S-products in terms of T-products is able rather simply to define an infinite set of functions $\varphi(x_1, \cdots, x_l|y_1, \cdots, y_m|z_1, \cdots, z_n)$ corresponding to l bosons, m fermions and n anti-fermions, whose asymptotic values determine transitions from free states to free states and which reduce to free-field wave functions if the coupling constants are zero. The author obtains an infinite system of coupled differential equations, and an equivalent system of integral equations, satisfied by these functions. He claims that his theory provides a framework for the exact discussion of all quantum mechanical systems. *A. J. Coleman.*

Freese, Ernst. Die Wellengleichungen der Quantenelektrodynamik. Acta Physica Austriaca **8**, 289–308 (1954).

The general theory of the paper reviewed above is applied to the following questions in quantum electrodynamics: the derivation of the equations used by Karplus and Kroll to discuss the Lamb shift; the decay time of an electron from an excited state; the derivation of the equation for positronium; bremsstrahlung. No essentially new results are achieved but the power of the author's theory is clearly demonstrated by the simple manner in which basic equations for these problems are obtained. Further, it is most satisfying to have a complete theory from which, by making well-defined approximations, it seems possible to obtain all previous generally accepted theories. *A. J. Coleman.*

Jauch, J. M. A note concerning the quantization of spinor fields. Helvetica Phys. Acta **27**, 89–98 (1954).

The author asserts that the spinor field quantized according to the method of Majorana provides a gegenbeispiel to Schwinger's claim [Physical Rev. (2) **82**, 914–927 (1951); these Rev. **13**, 520] that the Action Principle together with invariance under time-reversal determine the commutation rules. He is led to consider a new type of commutation rule which involves an arbitrary real number between 0 and 1. When this number is 0 the rules stated by Schwinger apply. The value 1 corresponds to the Majorana field. Values other than 0 can apply only to particles without electromagnetic interaction. *A. J. Coleman* (Toronto, Ont.).

Adirovič, È. I., and Podgoreckiĭ, M. I. On the interaction of microsystems with zero-point fluctuations of an electromagnetic field. Akad. Nauk SSSR. Žurnal Èksper. Teoret. Fiz. **26**, 150–152 (1954). (Russian)

An harmonic oscillator with radiation damping is in interaction with the zero-point fluctuations of the electromagnetic field. By a classical argument the author shows that the difference between the energy of a bound and a free particle due to this interaction is approximately $\frac{1}{2}h\nu$ in the case that c/v is large, where v is the frequency of a, the free undamped oscillator. The zero-point energy of a quantum oscillator is thus "explained" by a classical argument illustrating the important role played by the vacuum fluctuations. *A. J. Coleman* (Toronto, Ont.).

Edwards, S. F. A note on the divergence of the perturbation method in field theory. Philos. Mag. (7) **45**, 758–761 (1954).

A simplified model of a quantized field theory is discussed, differing from the complete theory by the omission of "closed loop" phenomena and of renormalization problems. It is proved that in the model the series expansion in powers of the coupling constant g^2 will in general diverge. The proof is much shorter and more transparent than the explicit divergence-calculations which have been given previously

[Thirring, Helvetica Phys. Acta **26**, 33–52 (1953); these Rev. **14**, 708].

The starting-point is a formula obtained by Edwards and Peierls [Proc. Roy. Soc. London. Ser. A. **224**, 24–33 (1954); these Rev. **15**, 1010] for the propagation-function $S(x-x')$ of a Fermi particle interacting with a quantized Bose field:

$$(1)\quad S(x-x')=\lim_{N\to\infty} R^N\int\cdots\int da_1\cdots da_N \times G(x,x',a_1,\cdots,a_N)\exp\left(-ig^{-2}\sum_1^N a_n^2\right).$$

Here R is a normalizing constant, the a_n are real parameters integrated from $-\infty$ to $+\infty$, and G is the propagation-function for the Fermi particle in an unquantized Bose field

$$(2)\qquad \Omega(x)=a_1\phi_1(x)+\cdots+a_N\phi_N(x),$$

where the ϕ_n are a suitable set of normal orthogonal functions. The function G can be calculated by the Fredholm method [Salam and Matthews, Physical Rev. (2) **90**, 690–695 (1953); these Rev. **15**, 82] in the form

$$(3)\qquad G=H(x,x',a_1,\cdots,a_N)/K(a_1,\cdots,a_N),$$

where H and K are entire functions of the a_n. Thus G will have a finite radius of convergence as a function of the a_n. It then follows from a theorem of G. N. Watson [see H. Jeffreys and B. Jeffreys, Methods of mathematical physics, Cambridge, 1946, chapter 17; these Rev. **8**, 447] that S given by (1) will be an asymptotic expansion diverging for every finite value of g. The relation thus disclosed, between the divergence of the power-series in the quantized theory and the singularity in the Fredholm solution of the unquantized theory, seems to be so simple and general that it may throw light also on the behavior of the full theory with renormalization. *F. J. Dyson* (Princeton, N. J.).

Rayski, Jerzy. On a regular field theory. III. Acta Phys. Polonica **13**, 95–114 (1954). (Russian summary)

[For parts I–II see same Acta **11**, 314–327 (1953); **13**, 15–28 (1954); these Rev. **15**, 82, 917.] The convergence of the author's non-local field theory [Hanus and Rayski, ibid. **12**, 181–193 (1953); these Rev. **15**, 587] is tested by computing to order e^2 the vacuum-to-vacuum matrix element of the reaction matrix K. The result is finite, and it is shown how the usual divergent results of the local theory arise from it when the form-factor tends to a delta-function and when the region of space-time considered tends to infinity. *F. J. Dyson* (Princeton, N. J.).

Kamefuchi, Susumu. On the structure of the interaction of the elementary particles. V. Interaction of the second kind and non-local interaction. Progress Theoret. Physics **11**, 273–287 (1954).

This is the sequel to a series of papers, of which the most recent is by the author and H. Umezawa [same journal **9**, 529–549 (1953); these Rev. **15**, 83]. In this paper the method of compensation of divergences, by putting counter-terms into the Lagrangian of the theory, is applied in detail to a meson-pair theory with derivatives in the interaction. Such a theory is not renormalizable in the ordinary sense, that is to say an infinite number of counter-terms are required. There is a good deal of arbitrariness in the results which are obtained, depending on the way in which the counter-terms are chosen. It is shown how this method is in a sense equivalent to using a theory with a non-local form-factor interaction. *F. J. Dyson* (Princeton, N. J.).

Ling, Daniel S., Jr. Expansion of wave packets. Physical Rev. (2) **96**, 216–217 (1954).

The Fourier coefficients of a wave packet are proved to be equal to the coefficients obtained when the wave packet is expanded in terms of a set of functions appropriate to a scattering problem. *Author's summary.*

Rubinow, S. I. Generalized variational principle for the scattering amplitude. Physical Rev. (2) **96**, 218–219 (1954).

The Schwinger variational principle in differential form for the S-wave phase shift has been generalized so as to be applicable to the entire scattering amplitude. *Author's summary.*

Petiau, Gérard. Sur la représentation des corpuscules en interaction avec des champs extérieurs par des fonctions d'ondes à singularités localisées mobiles le long des trajectoires. C. R. Acad. Sci. Paris **239**, 344–346 (1954).

Petiau, Gérard. Sur les fonctions d'ondes à singularités localisées mobiles le long des trajectoires représentant l'électron en interaction avec un champ magnétique constant. C. R. Acad. Sci. Paris **239**, 792–794 (1954).

Lévy, Maurice M. A covariant treatment of meson-nucleon scattering. Physical Rev. (2) **94**, 460–468 (1954).

All of the infinite diagrams associated with the meson-nucleon system are separated out and a closed expression is given for their contribution to the wave function after renormalization. This yields a method of avoiding to all orders the renormalization difficulties arising in the computation of phase-shifts and cross-sections. The contribution from finite diagrams remains as a series expansion. *H. C. Corben* (Los Angeles, Calf.).

Thermodynamics, Statistical Mechanics

Prigogine, I. Sur la théorie variationnelle des phénomènes irréversibles. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **40**, 471–483 (1954).

The equations for certain irreversible processes (thermal conduction and chemical reaction rates) are written in a form involving variational derivatives. From this is derived a volume integral, quadratic in the thermodynamic fluxes, which for a stationary non-equilibrium state has a minimum value consistent with constraints on the system. For systems near equilibrium, this minimum principle is equivalent to that requiring a minimum entropy production. *G. Newell* (Providence, R. I.).

Prigogine, I., et Mel, H. C. Sur la stabilité thermodynamique. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **40**, 588–599 (1954).

A study is made of some consequences of defining a system as thermodynamically stable if the entropy production caused by any perturbation of the state of the system is negative. A comparison is made between this thermodynamic stability and the stability of solutions of the phenomenological rate-equations for the decay of fluctuations from equilibrium. It is shown that the Onsager reciprocal relations must be true in order that the former stability will assure stability in the latter sense. *G. Newell* (Providence, R. I.).